

> restart :

## Computation of a classical bifurcation diagram for a first-order coupling via singular ODEs

We use ideas from the Vessiot theory of singular differential equations to derive a vector field such that the bifurcation paths are trajectories and the bifurcation points steady states. In this worksheet, we always have  $N=M=3$ ; all other parameters can be set in the worksheet. The bifurcation parameter is  $\beta$ .

> with(LinearAlgebra) : with(plots) : with(plottools) : with(VectorCalculus) : with(ListTools) :  
with(RootFinding) : with(ColorTools) :

> interface(warnlevel=0);  
refinlvl := 1;

$$\text{refinlvl} := 1 \quad (1)$$

> tolbl := Color([68, 119, 170]) :  
tolgr := Color([34, 136, 51]) :  
tolye := Color([204, 187, 68]) :  
tolpu := Color([170, 51, 119]) :  
tolre := Color([238, 102, 119]) :  
tolcy := Color([102, 204, 238]) :

### Build model.

We set the parameters to their standard values in the paper and use a linear coupling coefficient function.

> r := 0; o1 := 1; p1 := -1; q1 := 0; o2 := 1; p2 := -2; q2 := 3;

$$\begin{aligned} r &:= 0 \\ o1 &:= 1 \\ p1 &:= -1 \\ q1 &:= 0 \\ o2 &:= 1 \\ p2 &:= -2 \\ q2 &:= 3 \end{aligned} \quad (2)$$

> s := ( $\mu, x$ ) → ( $o1 \cdot x^2 + p1 \cdot x + q1 - \mu$ ) · ( $o2 \cdot x^2 + p2 \cdot x + q2 - \mu$ );  
collect(s( $\mu, x$ ), x);

$$s := (\mu, x) \mapsto (o1 \cdot x^2 + p1 \cdot x + q1 + (-\mu)) \cdot (o2 \cdot x^2 + p2 \cdot x + q2 + (-\mu)) \\ x^4 - 3x^3 + (-2\mu + 5)x^2 + (3\mu - 3)x - \mu(-\mu + 3) \quad (3)$$

> h := ( $\beta, \gamma, x$ ) → ( $\beta + \gamma \cdot x$ )

$$h := (\beta, \gamma, x) \mapsto \beta + \gamma \cdot x \quad (4)$$

>  $\sigma := y_1 + y_2 + y_3$

$$\sigma := y_1 + y_2 + y_3 \quad (5)$$

Define the considered vector field describing the evolution of the traits.

>  $f_1 := -y_1 \cdot \text{expand}(s(\mu, y_1)) + h(\beta, \gamma, y_1) \cdot \sigma$ ;

$$\begin{aligned}
f_2 &:= -y_2 \cdot \text{expand}(s(\mu, y_2)) + h(\beta, \gamma, y_2) \cdot \sigma; \\
f_3 &:= -y_3 \cdot \text{expand}(s(\mu, y_3)) + h(\beta, \gamma, y_3) \cdot \sigma; \\
f_1 &:= -y_1 (y_1^4 - 2\mu y_1^2 - 3y_1^3 + \mu^2 + 3\mu y_1 + 5y_1^2 - 3\mu - 3y_1) + (\gamma y_1 + \beta) (y_1 + y_2 + y_3) \\
f_2 &:= -y_2 (y_2^4 - 2\mu y_2^2 - 3y_2^3 + \mu^2 + 3\mu y_2 + 5y_2^2 - 3\mu - 3y_2) + (\gamma y_2 + \beta) (y_1 + y_2 + y_3) \\
f_3 &:= -y_3 (y_3^4 - 2\mu y_3^2 - 3y_3^3 + \mu^2 + 3\mu y_3 + 5y_3^2 - 3\mu - 3y_3) + (\gamma y_3 + \beta) (y_1 + y_2 + y_3) \quad (6)
\end{aligned}$$

Compute its Jacobians both with respect to the variables only and in addition with respect to the coupling parameter  $\beta$ .

$$\triangleright J := \text{simplify}(\text{Jacobian}([f_1, f_2, f_3], [y_1, y_2, y_3]));$$

$$JJ := \text{simplify}(\text{Jacobian}([f_1, f_2, f_3], [\beta, y_1, y_2, y_3]));$$

$$J :=$$

$$JJ := \begin{bmatrix} -5y_1^4 + 12y_1^3 + 3(-5 + 2\mu)y_1^2 + 2(3 - 3\mu + \gamma)y_1 - \mu^2 - \dots & \gamma y_2 + \beta & \dots \\ & \gamma y_3 + \beta & \dots \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} y_1 + y_2 + y_3 & -5y_1^4 + 12y_1^3 + 3(-5 + 2\mu)y_1^2 + 2(\dots \\ y_1 + y_2 + y_3 & \gamma y \dots \\ y_1 + y_2 + y_3 & \gamma y \dots \end{bmatrix}$$

$$\triangleright \det J := \text{Determinant}(J) :$$

Setting up vector field  $\mathbf{Y}$  generating the **projected Vessiot distribution** using the adjoint of the Jacobian and the Jacobian with respect to  $\beta$  alone.

$$\triangleright C := \text{Adjoint}(J) :$$

$$\triangleright M := \text{Jacobian}([f_1, f_2, f_3], [\beta]);$$

$$M := \begin{bmatrix} y_1 + y_2 + y_3 \\ y_1 + y_2 + y_3 \\ y_1 + y_2 + y_3 \end{bmatrix} \quad (8)$$

>  $b := -\text{expand}(C \cdot M) :$   
 >  $Y := [\text{det}J, b[1, 1], b[2, 1], b[3, 1]] :$

**We consider now only the case  $\mu=7/2$  and  $\gamma=0$ .**

>  $mg := \mu = 3.5, \gamma = 0;$

$mg := \mu = 3.5, \gamma = 0$

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>  $f0l_1 := \text{subs}(mg, f_1) : f0l_2 := \text{subs}(mg, f_2) : f0l_3 := \text{subs}(mg, f_3) :$   
 $J0l := \text{subs}(mg, J) : JJ0l := \text{subs}(mg, JJ) : \text{det}J0l := \text{subs}(mg, \text{det}J) :$   
 $Y0l := \text{subs}(mg, Y) : b0l := \text{subs}(mg, b) :$

We use **Y** to set up a **system** in a form suitable for integration with **dsolve**.

>  $\text{vars} := [\beta(t), y_1(t), y_2(t), y_3(t)] :$   
 $\text{trafo} := \beta = \beta(t), y_1 = y_1(t), y_2 = y_2(t), y_3 = y_3(t) :$   
 >  $Y0lt := \text{subs}(\text{trafo}, Y0l) :$   
 $\text{sys}Y0l := [\text{diff}(\beta(t), t) = Y0lt[1], \text{diff}(y_1(t), t) = Y0lt[2], \text{diff}(y_2(t), t) = Y0lt[3],$   
 $\text{diff}(y_3(t), t) = Y0lt[4]] :$   
 $\text{sys}Y0lm := [\text{diff}(\beta(t), t) = -Y0lt[1], \text{diff}(y_1(t), t) = -Y0lt[2], \text{diff}(y_2(t), t) = -Y0lt[3],$   
 $\text{diff}(y_3(t), t) = -Y0lt[4]] :$

**Compute numerically bifurcation points** (we round to five digits for better identification of identical  $\beta$ -values).

>  $\text{prec} := 1e-5;$   
 $\text{pround} := (l, p) \rightarrow \text{subsop}(2 = \text{Round}(\text{op}(2, l), p), l) :$   
 $\text{prec} := 0.00001$

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>  $\text{sys}0l := [f0l_1, f0l_2, f0l_3, \text{det}J0l] :$   
 >  $\text{st} := \text{time}(\ ) : \text{fsol}0l := \text{map}((l, p) \rightarrow \text{map}(\text{pround}, l, p), \text{Isolate}(\text{sys}0l, [\beta, y_1, y_2, y_3]), \text{prec});$   
 $\text{time}(\ ) - \text{st}; \text{nops}(\text{fsol}0l);$   
 $\text{fsol}0l := [[\beta = -9.18405, y_1 = -1.06774, y_2 = -1.06774, y_3 = 1.71852], [\beta = -9.18405, y_1$   
 $= -1.06774, y_2 = 1.71852, y_3 = -1.06774], [\beta = -9.18405, y_1 = 1.71852, y_2 = -1.06774, y_3$   
 $= -1.06774], [\beta = -2.98825, y_1 = -0.92548, y_2 = -0.92548, y_3 = 0.67641], [\beta$   
 $= -2.98825, y_1 = -0.92548, y_2 = 0.67641, y_3 = -0.92548], [\beta = -2.98825, y_1 = 0.67641, y_2$   
 $= -0.92548, y_3 = -0.92548], [\beta = -2.71785, y_1 = -1.06774, y_2 = -1.06774, y_3 = 0.72647],$   
 $[\beta = -2.71785, y_1 = -1.06774, y_2 = 0.72647, y_3 = -1.06774], [\beta = -2.71785, y_1 = 0.72647,$   
 $y_2 = -1.06774, y_3 = -1.06774], [\beta = -1.26490, y_1 = -0.93937, y_2 = -0.93937, y_3$   
 $= -0.93937], [\beta = -1.20230, y_1 = -1.10320, y_2 = -1.03073, y_3 = -1.03073], [\beta$   
 $= -1.20230, y_1 = -1.03073, y_2 = -1.10320, y_3 = -1.03073], [\beta = -1.20230, y_1$   
 $= -1.03073, y_2 = -1.03073, y_3 = -1.10320], [\beta = -1.19551, y_1 = -1.06774, y_2$   
 $= -1.06774, y_3 = -1.06774], [\beta = -0.14375, y_1 = -1.44047, y_2 = -0.12218, y_3 = 2.25065],$

$$\begin{aligned}
& [\beta = -0.14375, y_I = -1.44047, y_2 = 2.25065, y_3 = -0.12218], [\beta = -0.14375, y_I \\
& = -0.12218, y_2 = -1.44047, y_3 = 2.25065], [\beta = -0.14375, y_I = -0.12218, y_2 = 2.25065, y_3 \\
& = -1.44047], [\beta = -0.14375, y_I = 2.25065, y_2 = -1.44047, y_3 = -0.12218], [\beta \\
& = -0.14375, y_I = 2.25065, y_2 = -0.12218, y_3 = -1.44047], [\beta = -0.11604, y_I = -1.44048, \\
& y_2 = -0.11985, y_3 = 2.41529], [\beta = -0.11604, y_I = -1.44048, y_2 = 2.41529, y_3 = -0.11985], \\
& [\beta = -0.11604, y_I = -0.11985, y_2 = -1.44048, y_3 = 2.41529], [\beta = -0.11604, y_I \\
& = -0.11985, y_2 = 2.41529, y_3 = -1.44048], [\beta = -0.11604, y_I = 2.41529, y_2 = -1.44048, y_3 \\
& = -0.11985], [\beta = -0.11604, y_I = 2.41529, y_2 = -0.11985, y_3 = -1.44048], [\beta \\
& = -0.07827, y_I = -1.44650, y_2 = 2.33323, y_3 = 2.33323], [\beta = -0.07827, y_I = 2.33323, y_2 \\
& = -1.44650, y_3 = 2.33323], [\beta = -0.07827, y_I = 2.33323, y_2 = 2.33323, y_3 = -1.44650], [\beta \\
& = -0.07819, y_I = -1.44651, y_2 = 2.33667, y_3 = 2.33667], [\beta = -0.07819, y_I = 2.33667, y_2 \\
& = -1.44651, y_3 = 2.33667], [\beta = -0.07819, y_I = 2.33667, y_2 = 2.33667, y_3 = -1.44651], [\beta \\
& = -0.04933, y_I = -0.11909, y_2 = -0.11909, y_3 = 2.25077], [\beta = -0.04933, y_I = -0.11909, \\
& y_2 = 2.25077, y_3 = -0.11909], [\beta = -0.04933, y_I = 2.25077, y_2 = -0.11909, y_3 = -0.11909], \\
& [\beta = -0.04918, y_I = -0.11278, y_2 = -0.11278, y_3 = 2.25086], [\beta = -0.04918, y_I \\
& = -0.11278, y_2 = 2.25086, y_3 = -0.11278], [\beta = -0.04918, y_I = 2.25086, y_2 = -0.11278, y_3 \\
& = -0.11278], [\beta = -0.04561, y_I = -0.11846, y_2 = -0.11846, y_3 = 2.41526], [\beta \\
& = -0.04561, y_I = -0.11846, y_2 = 2.41526, y_3 = -0.11846], [\beta = -0.04561, y_I = 2.41526, y_2 \\
& = -0.11846, y_3 = -0.11846], [\beta = -0.04549, y_I = -0.11278, y_2 = -0.11278, y_3 = 2.41520], \\
& [\beta = -0.04549, y_I = -0.11278, y_2 = 2.41520, y_3 = -0.11278], [\beta = -0.04549, y_I = 2.41520, \\
& y_2 = -0.11278, y_3 = -0.11278], [\beta = -0.03601, y_I = 2.33429, y_2 = 2.33429, y_3 = 2.33429], [\beta \\
& = -0.03599, y_I = 2.33588, y_2 = 2.33588, y_3 = 2.33746], [\beta = -0.03599, y_I = 2.33588, y_2 \\
& = 2.33746, y_3 = 2.33588], [\beta = -0.03599, y_I = 2.33746, y_2 = 2.33588, y_3 = 2.33588], [\beta \\
& = -0.03599, y_I = 2.33667, y_2 = 2.33667, y_3 = 2.33667], [\beta = -0.02270, y_I = -0.11423, y_2 \\
& = 2.25085, y_3 = 2.25085], [\beta = -0.02270, y_I = 2.25085, y_2 = -0.11423, y_3 = 2.25085], [\beta \\
& = -0.02270, y_I = 2.25085, y_2 = 2.25085, y_3 = -0.11423], [\beta = -0.02188, y_I = -0.11416, y_2 \\
& = 2.25085, y_3 = 2.41520], [\beta = -0.02188, y_I = -0.11416, y_2 = 2.41520, y_3 = 2.25085], [\beta \\
& = -0.02188, y_I = 2.25085, y_2 = -0.11416, y_3 = 2.41520], [\beta = -0.02188, y_I = 2.25085, y_2 \\
& = 2.41520, y_3 = -0.11416], [\beta = -0.02188, y_I = 2.41520, y_2 = -0.11416, y_3 = 2.25085], [\beta
\end{aligned}$$

$$\begin{aligned}
&= -0.02188, y_l = 2.41520, y_2 = 2.25085, y_3 = -0.11416], [\beta = -0.02112, y_l = -0.11410, y_2 \\
&= 2.41520, y_3 = 2.41520], [\beta = -0.02112, y_l = 2.41520, y_2 = -0.11410, y_3 = 2.41520], [\beta \\
&= -0.02112, y_l = 2.41520, y_2 = 2.41520, y_3 = -0.11410], [\beta = 0.03328, y_l = -1.44049, y_2 \\
&= -1.44049, y_3 = -0.11068], [\beta = 0.03328, y_l = -1.44049, y_2 = -0.11068, y_3 = -1.44049], \\
&[\beta = 0.03328, y_l = -0.11068, y_2 = -1.44049, y_3 = -1.44049], [\beta = 0.05978, y_l = -1.44049, \\
&y_2 = -0.11278, y_3 = -0.11278], [\beta = 0.05978, y_l = -0.11278, y_2 = -1.44049, y_3 \\
&= -0.11278], [\beta = 0.05978, y_l = -0.11278, y_2 = -0.11278, y_3 = -1.44049], [\beta = 0.06006, y_l \\
&= -1.44048, y_2 = -0.10519, y_3 = -0.10519], [\beta = 0.06006, y_l = -0.10519, y_2 = -1.44048, \\
&y_3 = -0.10519], [\beta = 0.06006, y_l = -0.10519, y_2 = -0.10519, y_3 = -1.44048], [\beta = 0.29439, \\
&y_l = -0.11278, y_2 = -0.11278, y_3 = -0.11278], [\beta = 0.30297, y_l = -0.13194, y_2 \\
&= -0.09360, y_3 = -0.09360], [\beta = 0.30297, y_l = -0.09360, y_2 = -0.13194, y_3 = -0.09360], \\
&[\beta = 0.30297, y_l = -0.09360, y_2 = -0.09360, y_3 = -0.13194], [\beta = 0.45775, y_l = -1.44641, \\
&y_2 = -1.44641, y_3 = 2.34692], [\beta = 0.45775, y_l = -1.44641, y_2 = 2.34692, y_3 = -1.44641], [\beta \\
&= 0.45775, y_l = 2.34692, y_2 = -1.44641, y_3 = -1.44641], [\beta = 0.58333, y_l = 0., y_2 = 0., y_3 \\
&= 0.], [\beta = 0.85203, y_l = 1.19070, y_2 = 2.75188, y_3 = 2.75188], [\beta = 0.85203, y_l = 2.75188, y_2 \\
&= 1.19070, y_3 = 2.75188], [\beta = 0.85203, y_l = 2.75188, y_2 = 2.75188, y_3 = 1.19070], [\beta \\
&= 0.89039, y_l = -1.09206, y_2 = 2.69007, y_3 = 2.69007], [\beta = 0.89039, y_l = 2.69007, y_2 \\
&= -1.09206, y_3 = 2.69007], [\beta = 0.89039, y_l = 2.69007, y_2 = 2.69007, y_3 = -1.09206], [\beta \\
&= 1.09304, y_l = 1.24385, y_2 = 1.24385, y_3 = 2.75258], [\beta = 1.09304, y_l = 1.24385, y_2 = 2.75258, \\
&y_3 = 1.24385], [\beta = 1.09304, y_l = 2.75258, y_2 = 1.24385, y_3 = 1.24385], [\beta = 1.12303, y_l \\
&= 1.10099, y_2 = 1.10099, y_3 = 2.74760], [\beta = 1.12303, y_l = 1.10099, y_2 = 2.74760, y_3 \\
&= 1.10099], [\beta = 1.12303, y_l = 2.74760, y_2 = 1.10099, y_3 = 1.10099], [\beta = 1.15072, y_l \\
&= -1.09407, y_2 = 1.72031, y_3 = 2.69000], [\beta = 1.15072, y_l = -1.09407, y_2 = 2.69000, y_3 \\
&= 1.72031], [\beta = 1.15072, y_l = 1.72031, y_2 = -1.09407, y_3 = 2.69000], [\beta = 1.15072, y_l \\
&= 1.72031, y_2 = 2.69000, y_3 = -1.09407], [\beta = 1.15072, y_l = 2.69000, y_2 = -1.09407, y_3 \\
&= 1.72031], [\beta = 1.15072, y_l = 2.69000, y_2 = 1.72031, y_3 = -1.09407], [\beta = 1.53498, y_l \\
&= 1.24385, y_2 = 1.24385, y_3 = 1.24385], [\beta = 1.55488, y_l = 1.14717, y_2 = 1.14717, y_3 \\
&= 1.33898], [\beta = 1.55488, y_l = 1.14717, y_2 = 1.33898, y_3 = 1.14717], [\beta = 1.55488, y_l \\
&= 1.33898, y_2 = 1.14717, y_3 = 1.14717], [\beta = 1.62632, y_l = -1.09646, y_2 = 1.72065, y_3
\end{aligned}$$

$$\begin{aligned}
&= 1.72065], [\beta = 1.62632, y_1 = 1.72065, y_2 = -1.09646, y_3 = 1.72065], [\beta = 1.62632, y_1 \\
&= 1.72065, y_2 = 1.72065, y_3 = -1.09646], [\beta = 1.65141, y_1 = -1.12638, y_2 = 0.71549, y_3 \\
&= 2.68792], [\beta = 1.65141, y_1 = -1.12638, y_2 = 2.68792, y_3 = 0.71549], [\beta = 1.65141, y_1 \\
&= 0.71549, y_2 = -1.12638, y_3 = 2.68792], [\beta = 1.65141, y_1 = 0.71549, y_2 = 2.68792, y_3 \\
&= -1.12638], [\beta = 1.65141, y_1 = 2.68792, y_2 = -1.12638, y_3 = 0.71549], [\beta = 1.65141, y_1 \\
&= 2.68792, y_2 = 0.71549, y_3 = -1.12638], [\beta = 1.78659, y_1 = 0.85508, y_2 = 0.85508, y_3 \\
&= 0.85508], [\beta = 2.85883, y_1 = -1.14033, y_2 = 0.70941, y_3 = 1.73277], [\beta = 2.85883, y_1 \\
&= -1.14033, y_2 = 1.73277, y_3 = 0.70941], [\beta = 2.85883, y_1 = 0.70941, y_2 = -1.14033, y_3 \\
&= 1.73277], [\beta = 2.85883, y_1 = 0.70941, y_2 = 1.73277, y_3 = -1.14033], [\beta = 2.85883, y_1 \\
&= 1.73277, y_2 = -1.14033, y_3 = 0.70941], [\beta = 2.85883, y_1 = 1.73277, y_2 = 0.70941, y_3 \\
&= -1.14033], [\beta = 6.89987, y_1 = -1.06774, y_2 = -1.06774, y_3 = 2.69050], [\beta = 6.89987, y_1 \\
&= -1.06774, y_2 = 2.69050, y_3 = -1.06774], [\beta = 6.89987, y_1 = 2.69050, y_2 = -1.06774, y_3 \\
&= -1.06774], [\beta = 7.99935, y_1 = -0.82200, y_2 = 0.59820, y_3 = 0.59820], [\beta = 7.99935, y_1 \\
&= 0.59820, y_2 = -0.82200, y_3 = 0.59820], [\beta = 7.99935, y_1 = 0.59820, y_2 = 0.59820, y_3 \\
&= -0.82200]]
\end{aligned}$$

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We store the found bifurcation points in a **data file**.

```

> fname := FileTools:-JoinPath([ "Maple", "MathBiol", "Speciation",
    cat("BetaBifPointsN3M3Mu", sprintf("%4.2f", subs(mg, μ)), ".txt" ) ], base = homedir);
fd := fopen(fname, WRITE) :
fprintf(fd, "Bifurcation points computed by BetaBifDiagramN3M3.mw\n") :
fprintf(fd, "o1=%1f, p1=%1f, q1=%1f, o2=%1f, p2=%1f, q2=%1f, r=%1f, N=%1d, mu=\n",
    o1, p1, q1, o2, p2, q2, r, 3, subs(mg, μ)) :
fprintf(fd, "beta\t\t x1\t\t x2\t\t x3\n") :
    fname := "C:\Users\seiler\Maple\MathBiol\Speciation\BifPointsN3M3Mu3.50.txt"

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> **for** bf **in** fsol01 **do**

```
    fprintf(fd, "%+12.8ft %+12.8ft %+12.8ft %+12.8f\n", op(map(rhs, bf)) ) :
```

**end do:**

> fclose(fd);

We select the  $\beta$ -values of the bifurcation points and select those in the specified range.

```
> bifbeta := MakeUnique(map(l→rhs(l[1]), fsol01)) ; nops(bifbeta)
```

```
bifbeta := [ -9.18405, -2.98825, -2.71785, -1.26490, -1.20230, -1.19551, -0.14375,
    -0.11604, -0.07827, -0.07819, -0.04933, -0.04918, -0.04561, -0.04549, -0.03601,
    -0.03599, -0.02270, -0.02188, -0.02112, 0.03328, 0.05978, 0.06006, 0.29439, 0.30297,
```

0.45775, 0.58333, 0.85203, 0.89039, 1.09304, 1.12303, 1.15072, 1.53498, 1.55488, 1.62632,  
1.65141, 1.78659, 2.85883, 6.89987, 7.99935 ]

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```
>  $\beta_{\max} := 5; \beta_{\min} := -\beta_{\max};$ 
  bifbetarel := select( $x \rightarrow (x > \beta_{\min})$  and ( $x < \beta_{\max}$ ), bifbeta); nops(bifbetarel);
  fsol01rel := select( $x \rightarrow (rhs(x[1]) > \beta_{\min})$  and ( $rhs(x[1]) < \beta_{\max}$ ), fsol01) :
  bfpcolor := [ 0 $ nops(fsol01rel) ] :
```

$\beta_{\max} := 5$

$\beta_{\min} := -5$

*bifbetarel* := [ -2.98825, -2.71785, -1.26490, -1.20230, -1.19551, -0.14375, -0.11604,  
-0.07827, -0.07819, -0.04933, -0.04918, -0.04561, -0.04549, -0.03601, -0.03599,  
-0.02270, -0.02188, -0.02112, 0.03328, 0.05978, 0.06006, 0.29439, 0.30297, 0.45775,  
0.58333, 0.85203, 0.89039, 1.09304, 1.12303, 1.15072, 1.53498, 1.55488, 1.62632, 1.65141,  
1.78659, 2.85883 ]

36

(14)

Some auxiliary quantities for the plotting.

```
> proj12 := l → subsop(4 = NULL, l) :
  box := [  $\beta_{\min} \dots \beta_{\max}, -2 \dots 4, -2 \dots 4$  ];
  maxbox := [ -2 ..300, -20 ..20, -20 ..22 ];
  tend := 100;
```

*box* := [ -5 ..5, -2 ..4, -2 ..4 ]

*maxbox* := [ -2 ..300, -20 ..20, -20 ..22 ]

*tend* := 100

(15)

We generate a list of  $\beta$ -values lying before, after and between the chosen bifurcation values. The steady states at these  $\beta$ -values are used as initial points for computing pieces of the bifurcation paths.

```
> nbeta := nops(bifbetarel);
  betalist := [ 0 $ nbeta + 1 ] :
  checkbetalist := [ 0 $ nbeta + 1 ] :
```

*nbeta* := 36

(16)

```
> betalist[1] := bifbetarel[1] - 1 :
  betalist[-1] := bifbetarel[-1] + 1 :
> for i from 1 to nbeta - 1 do
  betalist[i + 1] := 0.5 · (bifbetarel[i] + bifbetarel[i + 1]);
end do:
  checkbetalist := [ op(map(m → [  $\beta(t) - m$ , halt ], betalist)), [  $\beta(t) - \beta_{\max}$ , halt] ] :
```

Analysis of the stationary point for values of  $\beta$  between two bifurcation points. For each such value, we print the total number of stationary points and a classification of these in form of a 3x3 matrix: the row index corresponds to the number of different levels, the first column corresponds to stable, the second column to unstable and the third column to non-hyperbolic stationary points.

```
> trajs := Array(1 ..nbeta + 1) :
```

```

ivpl := Array(1..nbeta + 1) :
sst := Array(1..nbeta + 1, 1..3, 1..3, 0) :
nsol := Array(1..nbeta + 1) :
for i from 1 to nbeta + 1 do
  fsolm := map( (l,p) → map(pround, l, p), Isolate( subs(β = betalist[i], [f0l1, f0l2, f0l3]), [y1,
    y2, y3] ), prec);
  nsol[i] := nops(fsolm);
  ivps := map(l → [β(0) = betalist[i], op(l)], map(l → subs(trafo, t = 0, l), fsolm) ) :
  ivpc := map(l → proj12(map(rhs, l)), ivps) :
  ivpl[i] := pointplot3d(ivpc, symbol = solidsphere, color = green, symbolsize = 4) :
  if i = 1 then
    intrange := -tend..0;
  elif i = nbeta + 1 then
    intrange := 0..tend;
  else
    intrange := -tend..tend;
  end if;
  trajs[i] := Array(1..nops(fsolm));
  checkbetai := subsop(i = NULL, checkbetalist);
  for j from 1 to nops(fsolm) do
    nvals := nops( { op(map(rhs, fsolm[j])) } );
    Jj := subs(op(ivps[j]), subs(trafo, t = 0, J01) );
    revsj := map(Re, Eigenvalues(Jj) );
    if mul(revsj) = 0 then
      tcol := "FireBrick";
      stab := 3;
    else
      tcol := "Black";
    end if;
    evsj := max(revsj);
    if evsj < 0 then
      tsty := solid;
      stab := 1;
    elif evsj > 0 then
      tsty := dot;
      stab := 2;
    else error "not yet handled";
    end if;
    if nvals = 3 and stab = 1 then
      print("stable 3-split: ", fsolm[j]);
    end if;
    sst[i, nvals, stab] := sst[i, nvals, stab] + 1;
    dsn := dsolve( [ op(sysY01), op(ivps[j]) ], numeric, vars, method = rkf45, range = intrange,
    events = [ [ Y01t[1], halt], op(checkbetai) ] ) :
    trajs[i][j] := odeplot(dsn, [β(t), y1(t), y2(t)], color = tcol, linestyle = tsty, thickness = 1,
    refine = refinelnvl, view = maxbox) :
  end do;

```



```

    print(betalist[i], nops(fsolm), sst[i]);
end do:

```

$$\begin{aligned}
 & -3.98825, 13, \begin{bmatrix} 1 & 0 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 & -2.853050, 19, \begin{bmatrix} 1 & 0 & 0 \\ 6 & 12 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 & -1.991375, 25, \begin{bmatrix} 1 & 0 & 0 \\ 9 & 9 & 0 \\ 0 & 6 & 0 \end{bmatrix} \\
 & -1.233600, 27, \begin{bmatrix} 1 & 2 & 0 \\ 9 & 9 & 0 \\ 0 & 6 & 0 \end{bmatrix} \\
 & -1.198905, 33, \begin{bmatrix} 1 & 2 & 0 \\ 9 & 15 & 0 \\ 0 & 6 & 0 \end{bmatrix} \\
 & -0.669630, 33, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 0 & 6 & 0 \end{bmatrix}
 \end{aligned}$$

$$-0.129895, 45, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 0 & 18 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.44002, y_2 = -0.07417, y_3 = 2.41805]$

"stable 3-split: ",  $[y_1 = -1.44002, y_2 = 2.41805, y_3 = -0.07417]$

"stable 3-split: ",  $[y_1 = -0.07417, y_2 = -1.44002, y_3 = 2.41805]$

"stable 3-split: ",  $[y_1 = -0.07417, y_2 = 2.41805, y_3 = -1.44002]$

"stable 3-split: ",  $[y_1 = 2.41805, y_2 = -1.44002, y_3 = -0.07417]$

"stable 3-split: ",  $[y_1 = 2.41805, y_2 = -0.07417, y_3 = -1.44002]$

$$-0.097155, 57, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 6 & 24 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43942, y_2 = -0.05427, y_3 = 2.42158]$

"stable 3-split: ",  $[y_1 = -1.43942, y_2 = 2.42158, y_3 = -0.05427]$

"stable 3-split: ",  $[y_1 = -0.05427, y_2 = -1.43942, y_3 = 2.42158]$

"stable 3-split: ",  $[y_1 = -0.05427, y_2 = 2.42158, y_3 = -1.43942]$

"stable 3-split: ",  $[y_1 = 2.42158, y_2 = -1.43942, y_3 = -0.05427]$

"stable 3-split: ",  $[y_1 = 2.42158, y_2 = -0.05427, y_3 = -1.43942]$

$$-0.078230, 63, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 21 & 0 \\ 6 & 24 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43892, y_2 = -0.04202, y_3 = 2.42434]$

"stable 3-split: ",  $[y_1 = -1.43892, y_2 = 2.42434, y_3 = -0.04202]$

"stable 3-split: ",  $[y_1 = -0.04202, y_2 = -1.43892, y_3 = 2.42434]$

"stable 3-split: ",  $[y_1 = -0.04202, y_2 = 2.42434, y_3 = -1.43892]$

"stable 3-split: ",  $[y_1 = 2.42434, y_2 = -1.43892, y_3 = -0.04202]$

"stable 3-split: ",  $[y_1 = 2.42434, y_2 = -0.04202, y_3 = -1.43892]$

$$-0.063760, 69, \begin{bmatrix} 2 & 1 & 0 \\ 12 & 18 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43840, y_2 = -0.03114, y_3 = 2.42712]$

"stable 3-split: ",  $[y_1 = -1.43840, y_2 = 2.42712, y_3 = -0.03114]$

"stable 3-split: ",  $[y_1 = -0.03114, y_2 = -1.43840, y_3 = 2.42712]$

"stable 3-split: ",  $[y_1 = -0.03114, y_2 = 2.42712, y_3 = -1.43840]$

"stable 3-split: ",  $[y_1 = 2.42712, y_2 = -1.43840, y_3 = -0.03114]$

"stable 3-split: ",  $[y_1 = 2.42712, y_2 = -0.03114, y_3 = -1.43840]$

$$-0.049255, 75, \begin{bmatrix} 2 & 1 & 0 \\ 12 & 24 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43833, y_2 = -0.02982, y_3 = 2.42748]$

"stable 3-split: ",  $[y_1 = -1.43833, y_2 = 2.42748, y_3 = -0.02982]$

"stable 3-split: ",  $[y_1 = -0.02982, y_2 = -1.43833, y_3 = 2.42748]$

"stable 3-split: ",  $[y_1 = -0.02982, y_2 = 2.42748, y_3 = -1.43833]$

"stable 3-split: ",  $[y_1 = 2.42748, y_2 = -1.43833, y_3 = -0.02982]$

"stable 3-split: ",  $[y_1 = 2.42748, y_2 = -0.02982, y_3 = -1.43833]$

$$-0.047395, 81, \begin{bmatrix} 2 & 1 & 0 \\ 12 & 24 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43826, y_2 = -0.02853, y_3 = 2.42783]$

"stable 3-split: ",  $[y_1 = -1.43826, y_2 = 2.42783, y_3 = -0.02853]$

"stable 3-split: ",  $[y_1 = -0.02853, y_2 = -1.43826, y_3 = 2.42783]$

"stable 3-split: ",  $[y_1 = -0.02853, y_2 = 2.42783, y_3 = -1.43826]$

"stable 3-split: ",  $[y_1 = 2.42783, y_2 = -1.43826, y_3 = -0.02853]$

"stable 3-split: ",  $[y_1 = 2.42783, y_2 = -0.02853, y_3 = -1.43826]$

$$-0.045550, 87, \begin{bmatrix} 2 & 1 & 0 \\ 12 & 30 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43808, y_2 = -0.02523, y_3 = 2.42875]$

"stable 3-split: ",  $[y_1 = -1.43808, y_2 = 2.42875, y_3 = -0.02523]$

"stable 3-split: ",  $[y_1 = -0.02523, y_2 = -1.43808, y_3 = 2.42875]$

"stable 3-split: ",  $[y_1 = -0.02523, y_2 = 2.42875, y_3 = -1.43808]$

"stable 3-split: ",  $[y_1 = 2.42875, y_2 = -1.43808, y_3 = -0.02523]$

"stable 3-split: ",  $[y_1 = 2.42875, y_2 = -0.02523, y_3 = -1.43808]$

$$-0.040750, 93, \begin{bmatrix} 2 & 1 & 0 \\ 15 & 27 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43790, y_2 = -0.02204, y_3 = 2.42966]$

"stable 3-split: ",  $[y_1 = -1.43790, y_2 = 2.42966, y_3 = -0.02204]$

"stable 3-split: ",  $[y_1 = -0.02204, y_2 = -1.43790, y_3 = 2.42966]$

"stable 3-split: ",  $[y_1 = -0.02204, y_2 = 2.42966, y_3 = -1.43790]$

"stable 3-split: ",  $[y_1 = 2.42966, y_2 = -1.43790, y_3 = -0.02204]$

"stable 3-split: ",  $[y_1 = 2.42966, y_2 = -0.02204, y_3 = -1.43790]$

$$-0.036000, 95, \begin{bmatrix} 2 & 3 & 0 \\ 15 & 27 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43765, y_2 = -0.01771, y_3 = 2.43093]$

"stable 3-split: ",  $[y_1 = -1.43765, y_2 = 2.43093, y_3 = -0.01771]$

"stable 3-split: ",  $[y_1 = -0.01771, y_2 = -1.43765, y_3 = 2.43093]$

"stable 3-split: ",  $[y_1 = -0.01771, y_2 = 2.43093, y_3 = -1.43765]$

"stable 3-split: ",  $[y_1 = 2.43093, y_2 = -1.43765, y_3 = -0.01771]$

"stable 3-split: ",  $[y_1 = 2.43093, y_2 = -0.01771, y_3 = -1.43765]$

$$-0.029345, 101, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 33 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43738, y_2 = -0.01326, y_3 = 2.43227]$

"stable 3-split: ",  $[y_1 = -1.43738, y_2 = 2.43227, y_3 = -0.01326]$

"stable 3-split: ",  $[y_1 = -0.01326, y_2 = -1.43738, y_3 = 2.43227]$

"stable 3-split: ",  $[y_1 = -0.01326, y_2 = 2.43227, y_3 = -1.43738]$

"stable 3-split: ",  $[y_1 = 2.43227, y_2 = -1.43738, y_3 = -0.01326]$

"stable 3-split: ",  $[y_1 = 2.43227, y_2 = -0.01326, y_3 = -1.43738]$

$$-0.022290, 107, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43735, y_2 = -0.01277, y_3 = 2.43242]$

"stable 3-split: ",  $[y_1 = -1.43735, y_2 = 2.43242, y_3 = -0.01277]$

"stable 3-split: ",  $[y_1 = -0.01277, y_2 = -1.43735, y_3 = 2.43242]$

"stable 3-split: ",  $[y_1 = -0.01277, y_2 = 2.43242, y_3 = -1.43735]$

"stable 3-split: ",  $[y_1 = 2.43242, y_2 = -1.43735, y_3 = -0.01277]$

"stable 3-split: ",  $[y_1 = 2.43242, y_2 = -0.01277, y_3 = -1.43735]$

$$-0.021500, 119, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43624, y_2 = 0.00344, y_3 = 2.43763]$

"stable 3-split: ",  $[y_1 = -1.43624, y_2 = 2.43763, y_3 = 0.00344]$

"stable 3-split: ",  $[y_1 = 0.00344, y_2 = -1.43624, y_3 = 2.43763]$

"stable 3-split: ",  $[y_1 = 0.00344, y_2 = 2.43763, y_3 = -1.43624]$

"stable 3-split: ",  $[y_1 = 2.43763, y_2 = -1.43624, y_3 = 0.00344]$

"stable 3-split: ",  $[y_1 = 2.43763, y_2 = 0.00344, y_3 = -1.43624]$

$$0.006080, 125, \begin{bmatrix} 3 & 2 & 0 \\ 18 & 42 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43453, y_2 = 0.02489, y_3 = 2.44512]$

"stable 3-split: ",  $[y_1 = -1.43453, y_2 = 2.44512, y_3 = 0.02489]$

"stable 3-split: ",  $[y_1 = 0.02489, y_2 = -1.43453, y_3 = 2.44512]$

"stable 3-split: ",  $[y_1 = 0.02489, y_2 = 2.44512, y_3 = -1.43453]$

"stable 3-split: ",  $[y_1 = 2.44512, y_2 = -1.43453, y_3 = 0.02489]$

"stable 3-split: ",  $[y_1 = 2.44512, y_2 = 0.02489, y_3 = -1.43453]$

$$0.046530, 119, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43394, y_2 = 0.03156, y_3 = 2.44756]$

"stable 3-split: ",  $[y_1 = -1.43394, y_2 = 2.44756, y_3 = 0.03156]$

"stable 3-split: ",  $[y_1 = 0.03156, y_2 = -1.43394, y_3 = 2.44756]$

"stable 3-split: ",  $[y_1 = 0.03156, y_2 = 2.44756, y_3 = -1.43394]$

"stable 3-split: ",  $[y_1 = 2.44756, y_2 = -1.43394, y_3 = 0.03156]$

"stable 3-split: ",  $[y_1 = 2.44756, y_2 = 0.03156, y_3 = -1.43394]$

$$0.059920, 113, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.42828, y_2 = 0.08420, y_3 = 2.46804]$

"stable 3-split: ",  $[y_1 = -1.42828, y_2 = 2.46804, y_3 = 0.08420]$

"stable 3-split: ",  $[y_1 = 0.08420, y_2 = -1.42828, y_3 = 2.46804]$

"stable 3-split: ",  $[y_1 = 0.08420, y_2 = 2.46804, y_3 = -1.42828]$

"stable 3-split: ",  $[y_1 = 2.46804, y_2 = -1.42828, y_3 = 0.08420]$

"stable 3-split: ",  $[y_1 = 2.46804, y_2 = 0.08420, y_3 = -1.42828]$

$$0.177225, 107, \begin{bmatrix} 3 & 2 & 0 \\ 12 & 36 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.42153, y_2 = 0.13252, y_3 = 2.48787]$

"stable 3-split: ",  $[y_1 = -1.42153, y_2 = 2.48787, y_3 = 0.13252]$

"stable 3-split: ",  $[y_1 = 0.13252, y_2 = -1.42153, y_3 = 2.48787]$

"stable 3-split: ",  $[y_1 = 0.13252, y_2 = 2.48787, y_3 = -1.42153]$

"stable 3-split: ",  $[y_1 = 2.48787, y_2 = -1.42153, y_3 = 0.13252]$

"stable 3-split: ",  $[y_1 = 2.48787, y_2 = 0.13252, y_3 = -1.42153]$

$$0.298680, 107, \begin{bmatrix} 3 & 2 & 0 \\ 12 & 36 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.41646, y_2 = 0.16314, y_3 = 2.50059]$

"stable 3-split: ",  $[y_1 = -1.41646, y_2 = 2.50059, y_3 = 0.16314]$

"stable 3-split: ",  $[y_1 = 0.16314, y_2 = -1.41646, y_3 = 2.50059]$

"stable 3-split: ",  $[y_1 = 0.16314, y_2 = 2.50059, y_3 = -1.41646]$

"stable 3-split: ",  $[y_1 = 2.50059, y_2 = -1.41646, y_3 = 0.16314]$

"stable 3-split: ",  $[y_1 = 2.50059, y_2 = 0.16314, y_3 = -1.41646]$

$$0.380360, 101, \begin{bmatrix} 3 & 2 & 0 \\ 12 & 30 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.40669, y_2 = 0.21380, y_3 = 2.52152]$

"stable 3-split: ",  $[y_1 = -1.40669, y_2 = 2.52152, y_3 = 0.21380]$

"stable 3-split: ",  $[y_1 = 0.21380, y_2 = -1.40669, y_3 = 2.52152]$

"stable 3-split: ",  $[y_1 = 0.21380, y_2 = 2.52152, y_3 = -1.40669]$

"stable 3-split: ",  $[y_1 = 2.52152, y_2 = -1.40669, y_3 = 0.21380]$

"stable 3-split: ",  $[y_1 = 2.52152, y_2 = 0.21380, y_3 = -1.40669]$

$$0.520540, 95, \begin{bmatrix} 3 & 2 & 0 \\ 9 & 27 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.39031, y_2 = 0.28342, y_3 = 2.54955]$

"stable 3-split: ",  $[y_1 = -1.39031, y_2 = 2.54955, y_3 = 0.28342]$

"stable 3-split: ",  $[y_1 = 0.28342, y_2 = -1.39031, y_3 = 2.54955]$

"stable 3-split: ",  $[y_1 = 0.28342, y_2 = 2.54955, y_3 = -1.39031]$

"stable 3-split: ",  $[y_1 = 2.54955, y_2 = -1.39031, y_3 = 0.28342]$

"stable 3-split: ",  $[y_1 = 2.54955, y_2 = 0.28342, y_3 = -1.39031]$

$$0.717680, 95, \begin{bmatrix} 3 & 2 & 0 \\ 9 & 27 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.37490, y_2 = 0.33797, y_3 = 2.57064]$

"stable 3-split: ",  $[y_1 = -1.37490, y_2 = 2.57064, y_3 = 0.33797]$

"stable 3-split: ",  $[y_1 = 0.33797, y_2 = -1.37490, y_3 = 2.57064]$

"stable 3-split: ",  $[y_1 = 0.33797, y_2 = 2.57064, y_3 = -1.37490]$

"stable 3-split: ",  $[y_1 = 2.57064, y_2 = -1.37490, y_3 = 0.33797]$

"stable 3-split: ",  $[y_1 = 2.57064, y_2 = 0.33797, y_3 = -1.37490]$

$$0.871210, 89, \begin{bmatrix} 3 & 2 & 0 \\ 6 & 24 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.36071, y_2 = 0.38184, y_3 = 2.58694]$

"stable 3-split: ",  $[y_1 = -1.36071, y_2 = 2.58694, y_3 = 0.38184]$

"stable 3-split: ",  $[y_1 = 0.38184, y_2 = -1.36071, y_3 = 2.58694]$

"stable 3-split: ",  $[y_1 = 0.38184, y_2 = 2.58694, y_3 = -1.36071]$

"stable 3-split: ",  $[y_1 = 2.58694, y_2 = -1.36071, y_3 = 0.38184]$

"stable 3-split: ",  $[y_1 = 2.58694, y_2 = 0.38184, y_3 = -1.36071]$



$$0.991715, 83, \begin{bmatrix} 3 & 2 & 0 \\ 3 & 21 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.34472, y_2 = 0.42573, y_3 = 2.60260]$

"stable 3-split: ",  $[y_1 = -1.34472, y_2 = 2.60260, y_3 = 0.42573]$

"stable 3-split: ",  $[y_1 = 0.42573, y_2 = -1.34472, y_3 = 2.60260]$

"stable 3-split: ",  $[y_1 = 0.42573, y_2 = 2.60260, y_3 = -1.34472]$

"stable 3-split: ",  $[y_1 = 2.60260, y_2 = -1.34472, y_3 = 0.42573]$

"stable 3-split: ",  $[y_1 = 2.60260, y_2 = 0.42573, y_3 = -1.34472]$

$$1.108035, 77, \begin{bmatrix} 3 & 2 & 0 \\ 3 & 21 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.34032, y_2 = 0.43693, y_3 = 2.60649]$

"stable 3-split: ",  $[y_1 = -1.34032, y_2 = 2.60649, y_3 = 0.43693]$

"stable 3-split: ",  $[y_1 = 0.43693, y_2 = -1.34032, y_3 = 2.60649]$

"stable 3-split: ",  $[y_1 = 0.43693, y_2 = 2.60649, y_3 = -1.34032]$

"stable 3-split: ",  $[y_1 = 2.60649, y_2 = -1.34032, y_3 = 0.43693]$

"stable 3-split: ",  $[y_1 = 2.60649, y_2 = 0.43693, y_3 = -1.34032]$

$$1.136875, 71, \begin{bmatrix} 3 & 2 & 0 \\ 0 & 18 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.30165, y_2 = 0.52244, y_3 = 2.63467]$

"stable 3-split: ",  $[y_1 = -1.30165, y_2 = 2.63467, y_3 = 0.52244]$

"stable 3-split: ",  $[y_1 = 0.52244, y_2 = -1.30165, y_3 = 2.63467]$

"stable 3-split: ",  $[y_1 = 0.52244, y_2 = 2.63467, y_3 = -1.30165]$

"stable 3-split: ",  $[y_1 = 2.63467, y_2 = -1.30165, y_3 = 0.52244]$

"stable 3-split: ",  $[y_1 = 2.63467, y_2 = 0.52244, y_3 = -1.30165]$

$$1.342850, 59, \begin{bmatrix} 3 & 2 & 0 \\ 0 & 18 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.23791, y_2 = 0.62441, y_3 = 2.66464]$

"stable 3-split: ",  $[y_1 = -1.23791, y_2 = 2.66464, y_3 = 0.62441]$

"stable 3-split: ",  $[y_1 = 0.62441, y_2 = -1.23791, y_3 = 2.66464]$

"stable 3-split: ",  $[y_1 = 0.62441, y_2 = 2.66464, y_3 = -1.23791]$

"stable 3-split: ",  $[y_1 = 2.66464, y_2 = -1.23791, y_3 = 0.62441]$

"stable 3-split: ",  $[y_1 = 2.66464, y_2 = 0.62441, y_3 = -1.23791]$

$$1.544930, 59, \begin{bmatrix} 3 & 2 & 0 \\ 0 & 18 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.21305, y_2 = 0.65357, y_3 = 2.67245]$

"stable 3-split: ",  $[y_1 = -1.21305, y_2 = 2.67245, y_3 = 0.65357]$

"stable 3-split: ",  $[y_1 = 0.65357, y_2 = -1.21305, y_3 = 2.67245]$

"stable 3-split: ",  $[y_1 = 0.65357, y_2 = 2.67245, y_3 = -1.21305]$

"stable 3-split: ",  $[y_1 = 2.67245, y_2 = -1.21305, y_3 = 0.65357]$

"stable 3-split: ",  $[y_1 = 2.67245, y_2 = 0.65357, y_3 = -1.21305]$

$$1.590600, 53, \begin{bmatrix} 3 & 2 & 0 \\ 0 & 12 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.16764, y_2 = 0.69333, y_3 = 2.68256]$

"stable 3-split: ",  $[y_1 = -1.16764, y_2 = 2.68256, y_3 = 0.69333]$

"stable 3-split: ",  $[y_1 = 0.69333, y_2 = -1.16764, y_3 = 2.68256]$

"stable 3-split: ",  $[y_1 = 0.69333, y_2 = 2.68256, y_3 = -1.16764]$

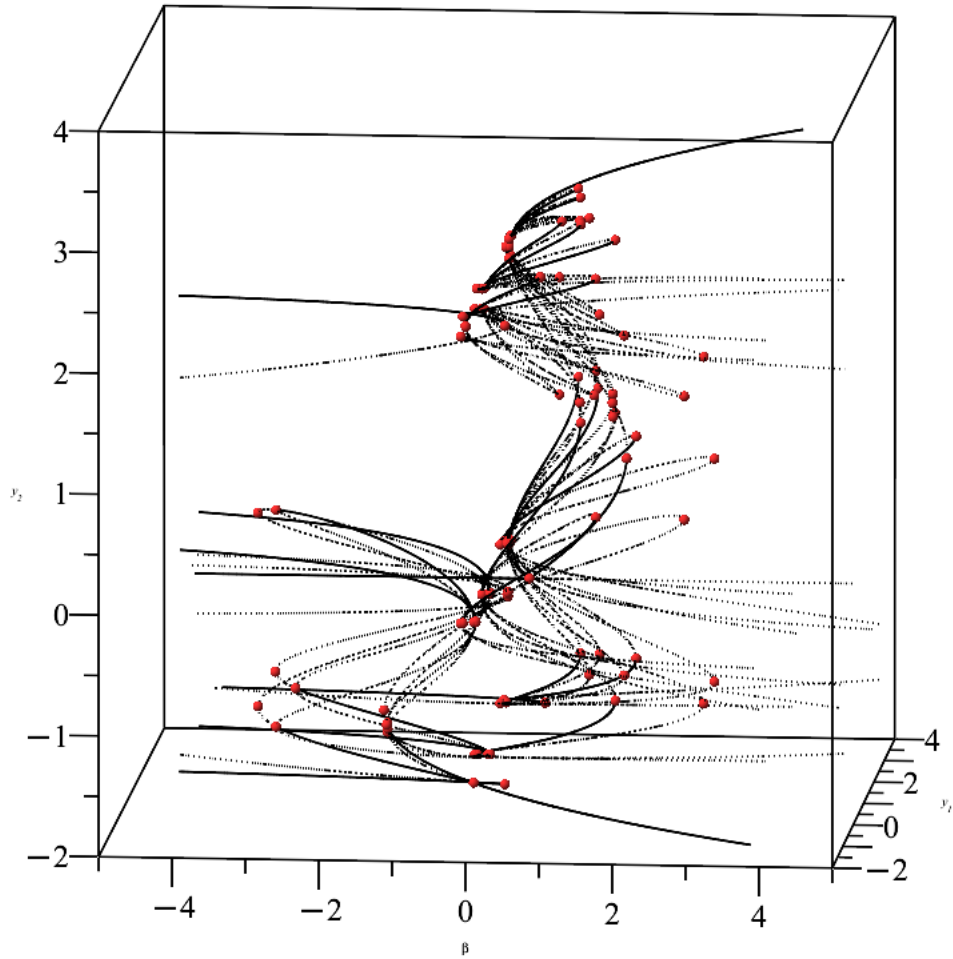
"stable 3-split: ",  $[y_1 = 2.68256, y_2 = -1.16764, y_3 = 0.69333]$

"stable 3-split: ",  $[y_1 = 2.68256, y_2 = 0.69333, y_3 = -1.16764]$

$$\begin{array}{l}
1.638865, 47, \begin{bmatrix} 3 & 2 & 0 \\ 0 & 6 & 0 \\ 6 & 30 & 0 \end{bmatrix} \\
1.719000, 35, \begin{bmatrix} 3 & 2 & 0 \\ 0 & 6 & 0 \\ 0 & 24 & 0 \end{bmatrix} \\
2.322710, 33, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 24 & 0 \end{bmatrix} \\
3.85883, 21, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 12 & 0 \end{bmatrix}
\end{array} \tag{17}$$

$$\begin{array}{l}
> \text{add}(\text{ArrayNumElems}(\text{trajs}[i]), i = 1 \dots \text{ArrayNumElems}(\text{trajs})) \\
\quad \quad \quad 2631
\end{array} \tag{18}$$

$$\begin{array}{l}
> \text{bifs} := \text{map}(l \rightarrow \text{proj12}(\text{map}(\text{rhs}, l)), \text{fsol01rel}) : \\
\quad \text{bifsp} := \text{pointplot3d}(\text{bifs}, \text{symbol} = \text{solidsphere}, \text{color} = \text{"FireBrick"}, \text{symbolsize} = 7) : \\
> \text{display}(\text{bifsp}, \text{seq}(\text{seq}(\text{trajs}[i][j], j = 1 \dots \text{ArrayNumElems}(\text{trajs}[i])), i = 1 \\
\quad \dots \text{ArrayNumElems}(\text{trajs})), \\
\quad \quad \text{labels} = [\text{'}\beta\text{'}, \text{'y}_1\text{'}, \text{'y}_2\text{'}, \text{view} = \text{box}, \text{orientation} = [-85, 80, 0], \text{lightmodel} = \text{light4})
\end{array}$$



>  $lth := 3$

$lth := 3$

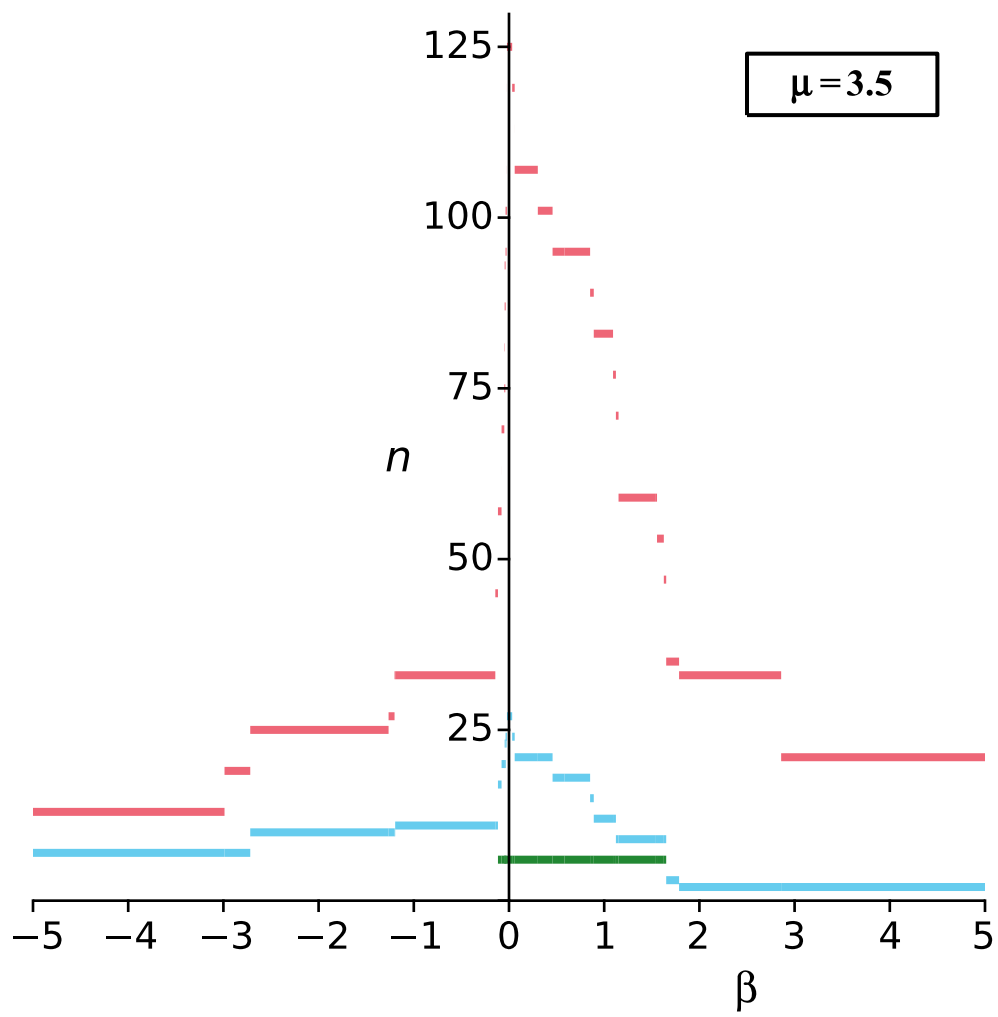
(19)

```
> nsollist := Array(1..nbeta + 1) :
  nstablist := Array(1..nbeta + 1) :
  nsymlist := Array(1..nbeta + 1) :
  nsollist[1] := line([ $\beta_{\min}$ , nsol[1]], [bifbetarel[1], nsol[1]], color = tolre, thickness = lth) :
  nstablist[1] := line([ $\beta_{\min}$ , add(sst[1, i, 1], i = 1..3)], [bifbetarel[1], add(sst[1, i, 1], i = 1..3)],
    color = tolcy, thickness = lth) :
  if sst[1, 3, 1]  $\neq$  0 then
```

```

    nsymlist[1] := line([ $\beta_{\min}$ , sst[1, 3, 1]], [bifbetarel[i], sst[1, 3, 1]], color = tolgr, thickness
    = lth) :
else
    nsymlist[1] := NULL :
end if:
nsollist[-1] := line([bifbetarel[-1], nsol[-1]], [ $\beta_{\max}$ , nsol[-1]], color = tolre, thickness
    = lth) :
nstablist[-1] := line([bifbetarel[-1], add(sst[-1, i, 1], i = 1 .. 3)], [ $\beta_{\max}$ , add(sst[-1, i, 1], i
    = 1 .. 3)], color = tolcy, thickness = lth) :
if sst[-1, 3, 1]  $\neq$  0 then
    nsymlist[-1] := line([bifbetarel[-1], sst[-1, 3, 1]], [ $\beta_{\max}$ , sst[-1, 3, 1]], color = tolgr,
    thickness = lth) :
else
    nsymlist[-1] := NULL :
end if:
for i from 2 to nbeta do
    nsollist[i] := line([bifbetarel[i-1], nsol[i]], [bifbetarel[i], nsol[i]], color = tolre, thickness
    = lth) :
    nstablist[i] := line([bifbetarel[i-1], add(sst[i, j, 1], j = 1 .. 3)], [bifbetarel[i], add(sst[i, j,
    1], j = 1 .. 3)], color = tolcy, thickness = lth) :
    if sst[i, 3, 1]  $\neq$  0 then
        nsymlist[i] := line([bifbetarel[i-1], sst[i, 3, 1]], [bifbetarel[i], sst[i, 3, 1]], color
        = tolgr, thickness = lth) :
    else
        nsymlist[i] := NULL;
    end if;
end do:
display(seq(nsollist[i], i = 1 .. nbeta + 1), seq(nstablist[i], i = 1 .. nbeta + 1), seq(nsymlist[i], i = 1
    .. nbeta + 1),
    textplot([3.5, 120, typeset( $\mu$  = subs(mg,  $\mu$ )), font = [Helvetica, bold, 14]]),
    rectangle([2.5, 115], [4.5, 124], filled = false),
    labels = [' $\beta$ ', 'n'], tickmarks = [11, 5], view = [ $\beta_{\min}$  ..  $\beta_{\max}$ , 0 .. 130],
    font = [Helvetica, roman, 14], labelfont = [Helvetica, roman, 16])

```



>