

> restart :

## Computation of a classical bifurcation diagram for a second-order coupling via singular ODEs

We use ideas from the Vessiot theory of singular differential equations to derive a vector field such that the bifurcation paths are trajectories and the bifurcation points steady states. In this worksheet, we always have  $N=M=3$ ; all other parameters can be set in the worksheet. The bifurcation parameter is  $\gamma$ .

> with(LinearAlgebra) : with(plots) : with(plottools) : with(VectorCalculus) : with(ListTools) :  
with(RootFinding) : with(ColorTools) :

> interface(warnlevel=0);  
refinlvl := 1;

$$\text{refinlvl} := 1 \quad (1)$$

> tolbl := Color([68, 119, 170]) :  
tolgr := Color([34, 136, 51]) :  
tolye := Color([204, 187, 68]) :  
tolpu := Color([170, 51, 119]) :  
tolre := Color([238, 102, 119]) :  
tolcy := Color([102, 204, 238]) :

### Build model.

We set the parameters to their standard values in the paper and use a linear coupling coefficient function.

> r := 0; o1 := 1; p1 := -1; q1 := 0; o2 := 1; p2 := -2; q2 := 3;

$$\begin{aligned} r &:= 0 \\ o1 &:= 1 \\ p1 &:= -1 \\ q1 &:= 0 \\ o2 &:= 1 \\ p2 &:= -2 \\ q2 &:= 3 \end{aligned} \quad (2)$$

> s := ( $\mu, x$ ) → ( $o1 \cdot x^2 + p1 \cdot x + q1 - \mu$ ) · ( $o2 \cdot x^2 + p2 \cdot x + q2 - \mu$ );  
collect(s( $\mu, x$ ), x);

$$s := (\mu, x) \mapsto (o1 \cdot x^2 + p1 \cdot x + q1 + (-\mu)) \cdot (o2 \cdot x^2 + p2 \cdot x + q2 + (-\mu)) \\ x^4 - 3x^3 + (-2\mu + 5)x^2 + (3\mu - 3)x - \mu(-\mu + 3) \quad (3)$$

> h := ( $\beta, \gamma, x$ ) → ( $\beta + \gamma \cdot x$ )

$$h := (\beta, \gamma, x) \mapsto \beta + \gamma x \quad (4)$$

>  $\sigma := y_1 + y_2 + y_3$

$$\sigma := y_1 + y_2 + y_3 \quad (5)$$

Define the considered vector field describing the evolution of the traits.

>  $f_1 := -y_1 \cdot \text{expand}(s(\mu, y_1)) + h(\beta, \gamma_0, y_1) \cdot \sigma$ ;

$$\begin{aligned}
f_2 &:= -y_2 \cdot \text{expand}(s(\mu, y_2)) + h(\beta, \gamma_0, y_2) \cdot \sigma; \\
f_3 &:= -y_3 \cdot \text{expand}(s(\mu, y_3)) + h(\beta, \gamma_0, y_3) \cdot \sigma; \\
f_1 &:= -y_1 (y_1^4 - 2\mu y_1^2 - 3y_1^3 + \mu^2 + 3\mu y_1 + 5y_1^2 - 3\mu - 3y_1) + (\gamma_0 y_1 + \beta) (y_1 + y_2 + y_3) \\
f_2 &:= -y_2 (y_2^4 - 2\mu y_2^2 - 3y_2^3 + \mu^2 + 3\mu y_2 + 5y_2^2 - 3\mu - 3y_2) + (\gamma_0 y_2 + \beta) (y_1 + y_2 + y_3) \\
f_3 &:= -y_3 (y_3^4 - 2\mu y_3^2 - 3y_3^3 + \mu^2 + 3\mu y_3 + 5y_3^2 - 3\mu - 3y_3) + (\gamma_0 y_3 + \beta) (y_1 + y_2 + y_3) \quad (6)
\end{aligned}$$

Compute its Jacobians both with respect to the variables only and in addition with respect to the coupling parameter  $\gamma$ .

>  $J := \text{simplify}(\text{Jacobian}([f_1, f_2, f_3], [y_1, y_2, y_3]));$

$JJ := \text{simplify}(\text{Jacobian}([f_1, f_2, f_3], [\gamma_0, y_1, y_2, y_3]));$

$J :=$

$$J := \begin{bmatrix} -5y_1^4 + 12y_1^3 + 3(-5 + 2\mu)y_1^2 + 2(3 - 3\mu + \gamma_0)y_1 - \mu^2 & \cdots & \cdots & \cdots \\ \cdots & y_2\gamma_0 + \beta & \cdots & \cdots \\ \cdots & \cdots & y_3\gamma_0 + \beta & \cdots \end{bmatrix} \quad (7)$$

$$JJ := \begin{bmatrix} \cdots - 5 + 2\mu y_1^2 + 2(3 - 3\mu + \gamma_0)y_1 - \mu^2 + 3\mu + (y_2 + y_3)\gamma_0 + \beta & \cdots & \cdots & \cdots \\ \cdots & y_2\gamma_0 + \beta & \cdots & \cdots \\ \cdots & \cdots & y_3\gamma_0 + \beta & \cdots \end{bmatrix}$$

>  $\det J := \text{Determinant}(J) :$

Setting up vector field  $\mathbf{Y}$  generating the **projected Vessiot distribution** using the adjoint of the Jacobian and the Jacobian with respect to  $\gamma$  alone.

>  $C := \text{Adjoint}(J) :$

>  $M := \text{Jacobian}([f_1, f_2, f_3], [\gamma_0]);$

$$M := \begin{bmatrix} y_1 (y_1 + y_2 + y_3) \\ y_2 (y_1 + y_2 + y_3) \\ y_3 (y_1 + y_2 + y_3) \end{bmatrix} \quad (8)$$

>  $b := -\text{expand}(C \cdot M) :$   
 >  $Y := [\text{det}J, b[1, 1], b[2, 1], b[3, 1]] :$

**We consider now only the case  $\mu=7/2$  and  $\beta=0$ .**

>  $mb := \mu = 3.5, \beta = 0;$

$$mb := \mu = 3.5, \beta = 0 \quad (9)$$

>  $f01_1 := \text{subs}(mb, f_1) : f01_2 := \text{subs}(mb, f_2) : f01_3 := \text{subs}(mb, f_3) :$   
 $J01 := \text{subs}(mb, J) : JJ01 := \text{subs}(mb, JJ) : \text{det}J01 := \text{subs}(mb, \text{det}J) :$   
 $Y01 := \text{subs}(mb, Y) : b01 := \text{subs}(mb, b) :$

We use **Y** to set up a **system** in a form suitable for integration with **dsolve**.

>  $\text{vars} := [\gamma(t), y_1(t), y_2(t), y_3(t)] :$   
 $\text{trafo} := \gamma_0 = \gamma(t), y_1 = y_1(t), y_2 = y_2(t), y_3 = y_3(t) :$   
 >  $Y01t := \text{subs}(\text{trafo}, Y01) :$   
 $\text{sys}Y01 := [\text{diff}(\gamma(t), t) = Y01t[1], \text{diff}(y_1(t), t) = Y01t[2], \text{diff}(y_2(t), t) = Y01t[3], \text{diff}(y_3(t), t) = Y01t[4]] :$   
 $\text{sys}Y01m := [\text{diff}(\gamma(t), t) = -Y01t[1], \text{diff}(y_1(t), t) = -Y01t[2], \text{diff}(y_2(t), t) = -Y01t[3], \text{diff}(y_3(t), t) = -Y01t[4]] :$

**Compute numerically bifurcation points** (we round to five digits for better identification of identical  $\gamma$  values).

>  $\text{prec} := 1e-5;$   
 $\text{pround} := (l, p) \rightarrow \text{subsop}(2 = \text{Round}(\text{op}(2, l), p), l) :$   
 $\text{prec} := 0.00001$

(10)

>  $\text{sys01} := [f01_1, f01_2, f01_3, \text{det}J01] :$   
 >  $\text{st} := \text{time}() : \text{fsol01} := \text{map}((l, p) \rightarrow \text{map}(\text{pround}, l, p), \text{Isolate}(\text{sys01}, [\gamma_0, y_1, y_2, y_3]), \text{prec}) ;$   
 $\text{time}() - \text{st}; \text{nops}(\text{fsol01});$

$\text{fsol01} := [[\gamma_0 = -2.28033, y_1 = -1.65573, y_2 = -1.65573, y_3 = 1.12674], [\gamma_0 = -2.28033, y_1 = -1.65573, y_2 = 1.12674, y_3 = -1.65573], [\gamma_0 = -2.28033, y_1 = 1.12674, y_2 = -1.65573, y_3 = -1.65573], [\gamma_0 = -1.14613, y_1 = -1.52688, y_2 = 0., y_3 = 0.], [\gamma_0 = -1.14613, y_1 = 0., y_2 = -1.52688, y_3 = 0.], [\gamma_0 = -1.14613, y_1 = 0., y_2 = 0., y_3 = -1.52688], [\gamma_0 = -0.57307, y_1 = -1.52688, y_2 = -1.52688, y_3 = 0.], [\gamma_0 = -0.57307, y_1 = -1.52688, y_2 = 0., y_3 = -1.52688], [\gamma_0 = -0.57307, y_1 = 0., y_2 = -1.52688, y_3 = -1.52688], [\gamma_0 = -0.16331, y_1 = -1.52688, y_2 = -1.52688, y_3 = -1.52688]]$

$$\begin{aligned}
&= -1.43019, y_2 = -0.23834, y_3 = 2.32588], [\gamma_0 = -0.16331, y_I = -1.43019, y_2 = 2.32588, y_3 \\
&= -0.23834], [\gamma_0 = -0.16331, y_I = -0.23834, y_2 = -1.43019, y_3 = 2.32588], [\gamma_0 \\
&= -0.16331, y_I = -0.23834, y_2 = 2.32588, y_3 = -1.43019], [\gamma_0 = -0.16331, y_I = 2.32588, y_2 \\
&= -1.43019, y_3 = -0.23834], [\gamma_0 = -0.16331, y_I = 2.32588, y_2 = -0.23834, y_3 = -1.43019], \\
&[\gamma_0 = -0.11990, y_I = -1.43017, y_2 = 0., y_3 = 2.32802], [\gamma_0 = -0.11990, y_I = -1.43017, y_2 \\
&= 2.32802, y_3 = 0.], [\gamma_0 = -0.11990, y_I = 0., y_2 = -1.43017, y_3 = 2.32802], [\gamma_0 = -0.11990, y_I \\
&= 0., y_2 = 2.32802, y_3 = -1.43017], [\gamma_0 = -0.11990, y_I = 2.32802, y_2 = -1.43017, y_3 = 0.], [\gamma_0 \\
&= -0.11990, y_I = 2.32802, y_2 = 0., y_3 = -1.43017], [\gamma_0 = -0.05821, y_I = -0.23842, y_2 \\
&= -0.23842, y_3 = 2.33132], [\gamma_0 = -0.05821, y_I = -0.23842, y_2 = 2.33132, y_3 = -0.23842], \\
&[\gamma_0 = -0.05821, y_I = 2.33132, y_2 = -0.23842, y_3 = -0.23842], [\gamma_0 = -0.05158, y_I \\
&= -0.23842, y_2 = 0., y_3 = 2.33164], [\gamma_0 = -0.05158, y_I = -0.23842, y_2 = 2.33164, y_3 = 0.], [\gamma_0 \\
&= -0.05158, y_I = 0., y_2 = -0.23842, y_3 = 2.33164], [\gamma_0 = -0.05158, y_I = 0., y_2 = 2.33164, y_3 \\
&= -0.23842], [\gamma_0 = -0.05158, y_I = 2.33164, y_2 = -0.23842, y_3 = 0.], [\gamma_0 = -0.05158, y_I \\
&= 2.33164, y_2 = 0., y_3 = -0.23842], [\gamma_0 = -0.04630, y_I = 0., y_2 = 0., y_3 = 2.33189], [\gamma_0 \\
&= -0.04630, y_I = 0., y_2 = 2.33189, y_3 = 0.], [\gamma_0 = -0.04630, y_I = 2.33189, y_2 = 0., y_3 = 0.], [\gamma_0 \\
&= -0.03340, y_I = -1.43016, y_2 = 2.33082, y_3 = 2.33082], [\gamma_0 = -0.03340, y_I = 2.33082, y_2 \\
&= -1.43016, y_3 = 2.33082], [\gamma_0 = -0.03340, y_I = 2.33082, y_2 = 2.33082, y_3 = -1.43016], [\gamma_0 \\
&= -0.03336, y_I = -1.43015, y_2 = 2.33429, y_3 = 2.33429], [\gamma_0 = -0.03336, y_I = 2.33429, y_2 \\
&= -1.43015, y_3 = 2.33429], [\gamma_0 = -0.03336, y_I = 2.33429, y_2 = 2.33429, y_3 = -1.43015], [\gamma_0 \\
&= -0.02440, y_I = -0.23842, y_2 = 2.33177, y_3 = 2.33177], [\gamma_0 = -0.02440, y_I = 2.33177, y_2 \\
&= -0.23842, y_3 = 2.33177], [\gamma_0 = -0.02440, y_I = 2.33177, y_2 = 2.33177, y_3 = -0.23842], [\gamma_0 \\
&= -0.02439, y_I = -0.23843, y_2 = 2.33429, y_3 = 2.33429], [\gamma_0 = -0.02439, y_I = 2.33429, y_2 \\
&= -0.23843, y_3 = 2.33429], [\gamma_0 = -0.02439, y_I = 2.33429, y_2 = 2.33429, y_3 = -0.23843], [\gamma_0 \\
&= -0.02315, y_I = 0., y_2 = 2.33189, y_3 = 2.33189], [\gamma_0 = -0.02315, y_I = 2.33189, y_2 = 0., y_3 \\
&= 2.33189], [\gamma_0 = -0.02315, y_I = 2.33189, y_2 = 2.33189, y_3 = 0.], [\gamma_0 = -0.02314, y_I = 0., y_2 \\
&= 2.33429, y_3 = 2.33429], [\gamma_0 = -0.02314, y_I = 2.33429, y_2 = 0., y_3 = 2.33429], [\gamma_0 \\
&= -0.02314, y_I = 2.33429, y_2 = 2.33429, y_3 = 0.], [\gamma_0 = -0.01543, y_I = 2.33189, y_2 = 2.33189, \\
&y_3 = 2.33189], [\gamma_0 = -0.01543, y_I = 2.33349, y_2 = 2.33349, y_3 = 2.33509], [\gamma_0 = -0.01543, y_I \\
&= 2.33349, y_2 = 2.33509, y_3 = 2.33349], [\gamma_0 = -0.01543, y_I = 2.33509, y_2 = 2.33349, y_3
\end{aligned}$$

$$\begin{aligned}
&= 2.33349], [\gamma_0 = -0.01543, y_I = 2.33429, y_2 = 2.33429, y_3 = 2.33429], [\gamma_0 = 0.20743, y_I \\
&= -1.43021, y_2 = -1.43021, y_3 = 2.34464], [\gamma_0 = 0.20743, y_I = -1.43021, y_2 = 2.34464, y_3 \\
&= -1.43021], [\gamma_0 = 0.20743, y_I = 2.34464, y_2 = -1.43021, y_3 = -1.43021], [\gamma_0 = 0.32134, y_I \\
&= 0., y_2 = 2.72297, y_3 = 2.72297], [\gamma_0 = 0.32134, y_I = 2.72297, y_2 = 0., y_3 = 2.72297], [\gamma_0 \\
&= 0.32134, y_I = 2.72297, y_2 = 2.72297, y_3 = 0.], [\gamma_0 = 0.38658, y_I = 0., y_2 = 1.80391, y_3 \\
&= 2.72297], [\gamma_0 = 0.38658, y_I = 0., y_2 = 2.72297, y_3 = 1.80391], [\gamma_0 = 0.38658, y_I = 1.80391, y_2 \\
&= 0., y_3 = 2.72297], [\gamma_0 = 0.38658, y_I = 1.80391, y_2 = 2.72297, y_3 = 0.], [\gamma_0 = 0.38658, y_I \\
&= 2.72297, y_2 = 0., y_3 = 1.80391], [\gamma_0 = 0.38658, y_I = 2.72297, y_2 = 1.80391, y_3 = 0.], [\gamma_0 \\
&= 0.48506, y_I = 0., y_2 = 1.80391, y_3 = 1.80391], [\gamma_0 = 0.48506, y_I = 1.80391, y_2 = 0., y_3 \\
&= 1.80391], [\gamma_0 = 0.48506, y_I = 1.80391, y_2 = 1.80391, y_3 = 0.], [\gamma_0 = 0.64268, y_I = 0., y_2 = 0., y_3 \\
&= 2.72297], [\gamma_0 = 0.64268, y_I = 0., y_2 = 2.72297, y_3 = 0.], [\gamma_0 = 0.64268, y_I = 2.72297, y_2 = 0., y_3 \\
&= 0.], [\gamma_0 = 0.79619, y_I = 0.77432, y_2 = 2.95685, y_3 = 2.95685], [\gamma_0 = 0.79619, y_I = 2.95685, y_2 \\
&= 0.77432, y_3 = 2.95685], [\gamma_0 = 0.79619, y_I = 2.95685, y_2 = 2.95685, y_3 = 0.77432], [\gamma_0 \\
&= 0.97012, y_I = 0., y_2 = 0., y_3 = 1.80391], [\gamma_0 = 0.97012, y_I = 0., y_2 = 1.80391, y_3 = 0.], [\gamma_0 \\
&= 0.97012, y_I = 1.80391, y_2 = 0., y_3 = 0.], [\gamma_0 = 1.14802, y_I = 0.85508, y_2 = 0.85508, y_3 \\
&= 2.95855], [\gamma_0 = 1.14802, y_I = 0.85508, y_2 = 2.95855, y_3 = 0.85508], [\gamma_0 = 1.14802, y_I \\
&= 2.95855, y_2 = 0.85508, y_3 = 0.85508], [\gamma_0 = 1.21041, y_I = 0.61381, y_2 = 0.61381, y_3 \\
&= 2.94306], [\gamma_0 = 1.21041, y_I = 0.61381, y_2 = 2.94306, y_3 = 0.61381], [\gamma_0 = 1.21041, y_I \\
&= 2.94306, y_2 = 0.61381, y_3 = 0.61381], [\gamma_0 = 1.34654, y_I = -0.93937, y_2 = -0.93937, y_3 \\
&= -0.93937], [\gamma_0 = 1.36213, y_I = -1.00278, y_2 = -0.87356, y_3 = -0.87356], [\gamma_0 = 1.36213, \\
&y_I = -0.87356, y_2 = -1.00278, y_3 = -0.87356], [\gamma_0 = 1.36213, y_I = -0.87356, y_2 \\
&= -0.87356, y_3 = -1.00278], [\gamma_0 = 1.43248, y_I = 0., y_2 = 0.71121, y_3 = 2.95311], [\gamma_0 \\
&= 1.43248, y_I = 0., y_2 = 2.95311, y_3 = 0.71121], [\gamma_0 = 1.43248, y_I = 0.71121, y_2 = 0., y_3 \\
&= 2.95311], [\gamma_0 = 1.43248, y_I = 0.71121, y_2 = 2.95311, y_3 = 0.], [\gamma_0 = 1.43248, y_I = 2.95311, y_2 \\
&= 0., y_3 = 0.71121], [\gamma_0 = 1.43248, y_I = 2.95311, y_2 = 0.71121, y_3 = 0.], [\gamma_0 = 1.46310, y_I \\
&= -1.52688, y_2 = 0., y_3 = 2.72297], [\gamma_0 = 1.46310, y_I = -1.52688, y_2 = 2.72297, y_3 = 0.], [\gamma_0 \\
&= 1.46310, y_I = 0., y_2 = -1.52688, y_3 = 2.72297], [\gamma_0 = 1.46310, y_I = 0., y_2 = 2.72297, y_3 \\
&= -1.52688], [\gamma_0 = 1.46310, y_I = 2.72297, y_2 = -1.52688, y_3 = 0.], [\gamma_0 = 1.46310, y_I \\
&= 2.72297, y_2 = 0., y_3 = -1.52688], [\gamma_0 = 1.52908, y_I = -0.69519, y_2 = -0.69519, y_3
\end{aligned}$$

$$\begin{aligned}
&= -0.69519], [\gamma_0 = 2.01981, y_1 = -0.93937, y_2 = -0.93937, y_3 = 0.], [\gamma_0 = 2.01981, y_1 \\
&= -0.93937, y_2 = 0., y_3 = -0.93937], [\gamma_0 = 2.01981, y_1 = 0., y_2 = -0.93937, y_3 = -0.93937], \\
&[\gamma_0 = 2.08938, y_1 = 0.85508, y_2 = 0.85508, y_3 = 0.85508], [\gamma_0 = 2.17255, y_1 = 0.65890, y_2 \\
&= 0.65890, y_3 = 1.05441], [\gamma_0 = 2.17255, y_1 = 0.65890, y_2 = 1.05441, y_3 = 0.65890], [\gamma_0 \\
&= 2.17255, y_1 = 1.05441, y_2 = 0.65890, y_3 = 0.65890], [\gamma_0 = 2.29362, y_1 = -0.69519, y_2 \\
&= -0.69519, y_3 = 0.], [\gamma_0 = 2.29362, y_1 = -0.69519, y_2 = 0., y_3 = -0.69519], [\gamma_0 = 2.29362, y_1 \\
&= 0., y_2 = -0.69519, y_3 = -0.69519], [\gamma_0 = 2.65954, y_1 = -1.65626, y_2 = 0.59474, y_3 \\
&= 2.94046], [\gamma_0 = 2.65954, y_1 = -1.65626, y_2 = 2.94046, y_3 = 0.59474], [\gamma_0 = 2.65954, y_1 \\
&= 0.59474, y_2 = -1.65626, y_3 = 2.94046], [\gamma_0 = 2.65954, y_1 = 0.59474, y_2 = 2.94046, y_3 \\
&= -1.65626], [\gamma_0 = 2.65954, y_1 = 2.94046, y_2 = -1.65626, y_3 = 0.59474], [\gamma_0 = 2.65954, y_1 \\
&= 2.94046, y_2 = 0.59474, y_3 = -1.65626], [\gamma_0 = 3.13406, y_1 = 0., y_2 = 0.85508, y_3 = 0.85508], \\
&[\gamma_0 = 3.13406, y_1 = 0.85508, y_2 = 0., y_3 = 0.85508], [\gamma_0 = 3.13406, y_1 = 0.85508, y_2 = 0.85508, y_3 \\
&= 0.], [\gamma_0 = 4.58724, y_1 = -0.69519, y_2 = 0., y_3 = 0.], [\gamma_0 = 4.58724, y_1 = 0., y_2 = -0.69519, y_3 \\
&= 0.], [\gamma_0 = 4.58724, y_1 = 0., y_2 = 0., y_3 = -0.69519], [\gamma_0 = 6.31699, y_1 = -1.52688, y_2 = 0., y_3 \\
&= 1.80391], [\gamma_0 = 6.31699, y_1 = -1.52688, y_2 = 1.80391, y_3 = 0.], [\gamma_0 = 6.31699, y_1 = 0., y_2 \\
&= -1.52688, y_3 = 1.80391], [\gamma_0 = 6.31699, y_1 = 0., y_2 = 1.80391, y_3 = -1.52688], [\gamma_0 \\
&= 6.31699, y_1 = 1.80391, y_2 = -1.52688, y_3 = 0.], [\gamma_0 = 6.31699, y_1 = 1.80391, y_2 = 0., y_3 \\
&= -1.52688], [\gamma_0 = 129.31886, y_1 = -1.66872, y_2 = 0.85508, y_3 = 0.85508], [\gamma_0 = 129.31886, \\
&y_1 = 0.85508, y_2 = -1.66872, y_3 = 0.85508], [\gamma_0 = 129.31886, y_1 = 0.85508, y_2 = 0.85508, y_3 \\
&= -1.66872]]
\end{aligned}$$

223.219

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(11)

We store the found bifurcation points in a **data file**.

```

> fname := FileTools:-JoinPath([ "Maple", "MathBiol", "Speciation",
    cat("GammaBifPointsN3M3Mu", sprintf("%4.2f", subs(mb, μ)), ".txt" ), base = homedir);
fd := fopen(fname, WRITE) :
fprintf(fd, "Bifurcation points computed by GammaaBifDiagramN3M3.mw\n") :
fprintf(fd, "o1=%1f, p1=%1f, q1=%1f, o2=%1f, p2=%1f, q2=%1f, r=%3f, N=%1d, mu=\n",
    o1, p1, q1, o2, p2, q2, r, 3, subs(mb, μ)) :
fprintf(fd, "beta\t\t x1\t\t x2\t\t x3\n") :
    fname := "C:\Users\seiler\Maple\MathBiol\Speciation\GammaBifPointsN3M3Mu3.50.txt" (12)

> for bf in fsol01 do
    fprintf(fd, "%+12.8ft %+12.8ft %+12.8ft %+12.8ft\n", op(map(rhs, bf))) :
end do:

```

> *fclose(fd);*

We select the  $\gamma$ -values of the bifurcation points and select those in the specified range.

> *bifgamma := MakeUnique(map(l→rhs(l[1]), fsol01)); nops(bifgamma)*  
*bifgamma := [ -2.28033, -1.14613, -0.57307, -0.16331, -0.11990, -0.05821, -0.05158,*  
*-0.04630, -0.03340, -0.03336, -0.02440, -0.02439, -0.02315, -0.02314, -0.01543,*  
*0.20743, 0.32134, 0.38658, 0.48506, 0.64268, 0.79619, 0.97012, 1.14802, 1.21041, 1.34654,*  
*1.36213, 1.43248, 1.46310, 1.52908, 2.01981, 2.08938, 2.17255, 2.29362, 2.65954, 3.13406,*  
*4.58724, 6.31699, 129.31886]*

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(13)

>  $\gamma_{\max} := 5; \gamma_{\min} := -\gamma_{\max};$   
*bifgammarel := select(x→(x >  $\gamma_{\min}$ ) and (x <  $\gamma_{\max}$ ), bifgamma); nops(bifgammarel);*  
*fsol01rel := select(x→(rhs(x[1]) >  $\gamma_{\min}$ ) and (rhs(x[1]) <  $\gamma_{\max}$ ), fsol01) :*  
*bfpcolor := [ 0 \$ nops(fsol01rel) ] :*

$\gamma_{\max} := 5$

$\gamma_{\min} := -5$

*bifgammarel := [ -2.28033, -1.14613, -0.57307, -0.16331, -0.11990, -0.05821,*  
*-0.05158, -0.04630, -0.03340, -0.03336, -0.02440, -0.02439, -0.02315, -0.02314,*  
*-0.01543, 0.20743, 0.32134, 0.38658, 0.48506, 0.64268, 0.79619, 0.97012, 1.14802, 1.21041,*  
*1.34654, 1.36213, 1.43248, 1.46310, 1.52908, 2.01981, 2.08938, 2.17255, 2.29362, 2.65954,*  
*3.13406, 4.58724]*

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(14)

Some auxiliary quantities for the plotting.

> *proj12 := l→subsop(4=NULL, l) :*  
*box := [ $\gamma_{\min} \dots \gamma_{\max}, -2 \dots 4, -2 \dots 4]$ ;*  
*maxbox := [ -2 ..300, -20 ..20, -20 ..22];*  
*tend := 100;*

*box := [ -5 ..5, -2 ..4, -2 ..4]*

*maxbox := [ -2 ..300, -20 ..20, -20 ..22]*

*tend := 100*

(15)

We generate a list of  $\gamma$ -values lying before, after and between the chosen bifurcation values. The steady states at these  $\gamma$ -values are used as initial points for computing pieces of the bifurcation paths.

> *ngamma := nops(bifgammarel);*  
*gammalist := [ 0 \$ ngamma + 1 ] :*  
*checkgammalist := [ 0 \$ ngamma + 1 ] :*

*ngamma := 36*

(16)

> *gammalist[1] := bifgammarel[1] - 1 :*  
*gammalist[-1] := bifgammarel[-1] + 1 :*  
 > **for** *i* **from** 1 **to** *ngamma - 1* **do**  
   *gammalist[i + 1] := 0.5 · (bifgammarel[i] + bifgammarel[i + 1]);*  
**end do;**

$checkgammalist := [op(map(m \rightarrow [\gamma(t) - m, halt], gammalist)), [\gamma(t) - \gamma_{\max}, halt]] :$

Analysis of the stationary point for values of  $\gamma$  between two bifurcation points. For each such value, we print the total number of stationary points and a classification of these in form of a 3x3 matrix: the row index corresponds to the number of different levels, the first column corresponds to stable, the second column to unstable and the third column to non-hyperbolic stationary points.

```

> trajs := Array(1 ..ngamma + 1) :
ivpl := Array(1 ..ngamma + 1) :
sst := Array(1 ..ngamma + 1, 1 ..3, 1 ..3, 0) :
nsol := Array(1 ..ngamma + 1) :
for i from 1 to ngamma + 1 do
  fsolm := map((l, p) → map(pround, l, p), Isolate(subs(γ0 = gammalist[i], [f0l1, f0l2, f0l3]),
    [y1, y2, y3]), prec);
  nsol[i] := nops(fsolm);
  ivps := map(l → [γ(0) = gammalist[i], op(l)], map(l → subs(trafo, t = 0, l), fsolm)) :
  ivpc := map(l → proj12(map(rhs, l)), ivps) :
  ivpl[i] := pointplot3d(ivpc, symbol = solidsphere, color = green, symbolsize = 4) :
  if i = 1 then
    intrange := -tend ..0;
  elif i = ngamma + 1 then
    intrange := 0 ..tend;
  else
    intrange := -tend ..tend;
  end if;
  trajs[i] := Array(1 ..nsol[i]);
  checkgammali := subsop(i = NULL, checkgammalist);
  for j from 1 to nsol[i] do
    nvals := nops({op(map(rhs, fsolm[j]))});
    Jj := subs(op(ivps[j]), subs(trafo, t = 0, J01));
    revsj := map(Re, Eigenvalues(Jj));
    if mul(revsj) = 0 then
      tcol := "FireBrick";
      stab := 3;
    else
      tcol := "Black";
    end if;
    evsj := max(revsj);
    if evsj < 0 then
      tsty := solid;
      stab := 1;
    elif evsj > 0 then
      tsty := dot;
      stab := 2;
    else error "not yet handled";
    end if;
    if nvals = 3 and stab = 1 then
      print("stable 3-split: ", fsolm[j]);
    end if;
  end for;

```



```

    sst[i, nvals, stab] := sst[i, nvals, stab] + 1;
    dsn := dsolve([op(sysY01), op(ivps[j])], numeric, vars, method=rkf45, range=inrange,
events = [[Y01t[1], halt], op(checkgammai)]) :
    trajs[i][j] := odeplot(dsn, [gamma(t), y1(t), y2(t)], color=tcol, linestyle=tsty, thickness=1,
refine=refinelvl, view=maxbox) :
end do:
print(gamalist[i], nsol[i], sst[i]);
end do:

```

$$-3.28033, 27, \begin{bmatrix} 2 & 1 & 0 \\ 6 & 12 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$-1.713230, 33, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$-0.859600, 33, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$-0.368190, 33, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$-0.141605, 45, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 0 & 18 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43153, y_2 = 0., y_3 = 2.38261]$

"stable 3-split: ",  $[y_1 = -1.43153, y_2 = 2.38261, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43153, y_3 = 2.38261]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.38261, y_3 = -1.43153]$

"stable 3-split: ",  $[y_1 = 2.38261, y_2 = -1.43153, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.38261, y_2 = 0., y_3 = -1.43153]$

$$-0.089055, 57, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 15 & 0 \\ 6 & 24 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43337, y_2 = 0., y_3 = 2.40759]$

"stable 3-split: ",  $[y_1 = -1.43337, y_2 = 2.40759, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43337, y_3 = 2.40759]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.40759, y_3 = -1.43337]$

"stable 3-split: ",  $[y_1 = 2.40759, y_2 = -1.43337, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.40759, y_2 = 0., y_3 = -1.43337]$

$$-0.054895, 63, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 21 & 0 \\ 6 & 24 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43370, y_2 = 0., y_3 = 2.41119]$

"stable 3-split: ",  $[y_1 = -1.43370, y_2 = 2.41119, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43370, y_3 = 2.41119]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.41119, y_3 = -1.43370]$

"stable 3-split: ",  $[y_1 = 2.41119, y_2 = -1.43370, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.41119, y_2 = 0., y_3 = -1.43370]$

$$-0.048940, 75, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 21 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43421, y_2 = 0., y_3 = 2.41643]$

"stable 3-split: ",  $[y_1 = -1.43421, y_2 = 2.41643, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43421, y_3 = 2.41643]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.41643, y_3 = -1.43421]$

"stable 3-split: ",  $[y_1 = 2.41643, y_2 = -1.43421, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.41643, y_2 = 0., y_3 = -1.43421]$

$$-0.039850, 81, \begin{bmatrix} 2 & 1 & 0 \\ 12 & 24 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43457, y_2 = 0., y_3 = 2.41997]$

"stable 3-split: ",  $[y_1 = -1.43457, y_2 = 2.41997, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43457, y_3 = 2.41997]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.41997, y_3 = -1.43457]$

"stable 3-split: ",  $[y_1 = 2.41997, y_2 = -1.43457, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.41997, y_2 = 0., y_3 = -1.43457]$

$$-0.033380, 87, \begin{bmatrix} 2 & 1 & 0 \\ 12 & 30 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43483, y_2 = 0., y_3 = 2.42236]$

"stable 3-split: ",  $[y_1 = -1.43483, y_2 = 2.42236, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43483, y_3 = 2.42236]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.42236, y_3 = -1.43483]$

"stable 3-split: ",  $[y_1 = 2.42236, y_2 = -1.43483, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.42236, y_2 = 0., y_3 = -1.43483]$

$$-0.028880, 93, \begin{bmatrix} 2 & 1 & 0 \\ 15 & 27 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43508, y_2 = 0., y_3 = 2.42469]$

"stable 3-split: ",  $[y_1 = -1.43508, y_2 = 2.42469, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43508, y_3 = 2.42469]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.42469, y_3 = -1.43508]$

"stable 3-split: ",  $[y_1 = 2.42469, y_2 = -1.43508, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.42469, y_2 = 0., y_3 = -1.43508]$

$$-0.024395, 99, \begin{bmatrix} 2 & 1 & 0 \\ 15 & 33 & 0 \\ 6 & 42 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43512, y_2 = 0., y_3 = 2.42501]$

"stable 3-split: ",  $[y_1 = -1.43512, y_2 = 2.42501, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43512, y_3 = 2.42501]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.42501, y_3 = -1.43512]$

"stable 3-split: ",  $[y_1 = 2.42501, y_2 = -1.43512, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.42501, y_2 = 0., y_3 = -1.43512]$

$$-0.023770, 105, \begin{bmatrix} 2 & 1 & 0 \\ 15 & 33 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43516, y_2 = 0., y_3 = 2.42533]$

"stable 3-split: ",  $[y_1 = -1.43516, y_2 = 2.42533, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43516, y_3 = 2.42533]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.42533, y_3 = -1.43516]$

"stable 3-split: ",  $[y_1 = 2.42533, y_2 = -1.43516, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.42533, y_2 = 0., y_3 = -1.43516]$

$$-0.023145, 111, \begin{bmatrix} 2 & 1 & 0 \\ 15 & 39 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.43538, y_2 = 0., y_3 = 2.42727]$

"stable 3-split: ",  $[y_1 = -1.43538, y_2 = 2.42727, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.43538, y_3 = 2.42727]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.42727, y_3 = -1.43538]$

"stable 3-split: ",  $[y_1 = 2.42727, y_2 = -1.43538, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.42727, y_2 = 0., y_3 = -1.43538]$

$$-0.019285, 117, \begin{bmatrix} 2 & 1 & 0 \\ 18 & 36 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.44221, y_2 = 0., y_3 = 2.47413]$

"stable 3-split: ",  $[y_1 = -1.44221, y_2 = 2.47413, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.44221, y_3 = 2.47413]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.47413, y_3 = -1.44221]$

"stable 3-split: ",  $[y_1 = 2.47413, y_2 = -1.44221, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.47413, y_2 = 0., y_3 = -1.44221]$

$$0.096000, 125, \begin{bmatrix} 3 & 2 & 0 \\ 18 & 42 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.45262, y_2 = 0., y_3 = 2.52356]$

"stable 3-split: ",  $[y_1 = -1.45262, y_2 = 2.52356, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.45262, y_3 = 2.52356]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.52356, y_3 = -1.45262]$

"stable 3-split: ",  $[y_1 = 2.52356, y_2 = -1.45262, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.52356, y_2 = 0., y_3 = -1.45262]$

$$0.264385, 119, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.45824, y_2 = 0., y_3 = 2.54531]$

"stable 3-split: ",  $[y_1 = -1.45824, y_2 = 2.54531, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.45824, y_3 = 2.54531]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.54531, y_3 = -1.45824]$

"stable 3-split: ",  $[y_1 = 2.54531, y_2 = -1.45824, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.54531, y_2 = 0., y_3 = -1.45824]$

$$0.353960, 119, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.46340, y_2 = 0., y_3 = 2.56342]$

"stable 3-split: ",  $[y_1 = -1.46340, y_2 = 2.56342, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.46340, y_3 = 2.56342]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.56342, y_3 = -1.46340]$

"stable 3-split: ",  $[y_1 = 2.56342, y_2 = -1.46340, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.56342, y_2 = 0., y_3 = -1.46340]$

$$0.435820, 119, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.47149, y_2 = 0., y_3 = 2.58916]$

"stable 3-split: ",  $[y_1 = -1.47149, y_2 = 2.58916, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.47149, y_3 = 2.58916]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.58916, y_3 = -1.47149]$

"stable 3-split: ",  $[y_1 = 2.58916, y_2 = -1.47149, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.58916, y_2 = 0., y_3 = -1.47149]$

$$0.563870, 119, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.48129, y_2 = 0., y_3 = 2.61723]$

"stable 3-split: ",  $[y_1 = -1.48129, y_2 = 2.61723, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.48129, y_3 = 2.61723]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.61723, y_3 = -1.48129]$

"stable 3-split: ",  $[y_1 = 2.61723, y_2 = -1.48129, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.61723, y_2 = 0., y_3 = -1.48129]$

$$0.719435, 119, \begin{bmatrix} 3 & 2 & 0 \\ 15 & 39 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.49153, y_2 = 0., y_3 = 2.64392]$

"stable 3-split: ",  $[y_1 = -1.49153, y_2 = 2.64392, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.49153, y_3 = 2.64392]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.64392, y_3 = -1.49153]$

"stable 3-split: ",  $[y_1 = 2.64392, y_2 = -1.49153, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.64392, y_2 = 0., y_3 = -1.49153]$

$$0.883155, 113, \begin{bmatrix} 3 & 2 & 0 \\ 12 & 36 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.50242, y_2 = 0., y_3 = 2.67008]$

"stable 3-split: ",  $[y_1 = -1.50242, y_2 = 2.67008, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.50242, y_3 = 2.67008]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.67008, y_3 = -1.50242]$

"stable 3-split: ",  $[y_1 = 2.67008, y_2 = -1.50242, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.67008, y_2 = 0., y_3 = -1.50242]$

$$1.059070, 113, \begin{bmatrix} 3 & 2 & 0 \\ 12 & 36 & 0 \\ 6 & 54 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.50978, y_2 = 0., y_3 = 2.68674]$

"stable 3-split: ",  $[y_1 = -1.50978, y_2 = 2.68674, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.50978, y_3 = 2.68674]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.68674, y_3 = -1.50978]$

"stable 3-split: ",  $[y_1 = 2.68674, y_2 = -1.50978, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.68674, y_2 = 0., y_3 = -1.50978]$

$$1.179215, 107, \begin{bmatrix} 3 & 2 & 0 \\ 12 & 36 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.51580, y_2 = 0., y_3 = 2.69987]$

"stable 3-split: ",  $[y_1 = -1.51580, y_2 = 2.69987, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.51580, y_3 = 2.69987]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.69987, y_3 = -1.51580]$

"stable 3-split: ",  $[y_1 = 2.69987, y_2 = -1.51580, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.69987, y_2 = 0., y_3 = -1.51580]$

$$1.278475, 101, \begin{bmatrix} 3 & 2 & 0 \\ 9 & 33 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.52037, y_2 = 0., y_3 = 2.70955]$

"stable 3-split: ",  $[y_1 = -1.52037, y_2 = 2.70955, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.52037, y_3 = 2.70955]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.70955, y_3 = -1.52037]$

"stable 3-split: ",  $[y_1 = 2.70955, y_2 = -1.52037, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.70955, y_2 = 0., y_3 = -1.52037]$

$$1.354335, 101, \begin{bmatrix} 2 & 3 & 0 \\ 9 & 33 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.52295, y_2 = 0., y_3 = 2.71492]$

"stable 3-split: ",  $[y_1 = -1.52295, y_2 = 2.71492, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.52295, y_3 = 2.71492]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.71492, y_3 = -1.52295]$

"stable 3-split: ",  $[y_1 = 2.71492, y_2 = -1.52295, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.71492, y_2 = 0., y_3 = -1.52295]$



$$1.397305, 95, \begin{bmatrix} 2 & 3 & 0 \\ 9 & 27 & 0 \\ 6 & 48 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.52597, y_2 = 0., y_3 = 2.72111]$

"stable 3-split: ",  $[y_1 = -1.52597, y_2 = 2.72111, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 0., y_2 = -1.52597, y_3 = 2.72111]$

"stable 3-split: ",  $[y_1 = 0., y_2 = 2.72111, y_3 = -1.52597]$

"stable 3-split: ",  $[y_1 = 2.72111, y_2 = -1.52597, y_3 = 0.]$

"stable 3-split: ",  $[y_1 = 2.72111, y_2 = 0., y_3 = -1.52597]$

$$1.447790, 83, \begin{bmatrix} 2 & 3 & 0 \\ 9 & 27 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.52937, y_2 = 0.00722, y_3 = 2.72801]$

"stable 3-split: ",  $[y_1 = -1.52937, y_2 = 2.72801, y_3 = 0.00722]$

"stable 3-split: ",  $[y_1 = 0.00722, y_2 = -1.52937, y_3 = 2.72801]$

"stable 3-split: ",  $[y_1 = 0.00722, y_2 = 2.72801, y_3 = -1.52937]$

"stable 3-split: ",  $[y_1 = 2.72801, y_2 = -1.52937, y_3 = 0.00722]$

"stable 3-split: ",  $[y_1 = 2.72801, y_2 = 0.00722, y_3 = -1.52937]$

$$1.496090, 83, \begin{bmatrix} 2 & 3 & 0 \\ 9 & 27 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.55136, y_2 = 0.07426, y_3 = 2.77032]$

"stable 3-split: ",  $[y_1 = -1.55136, y_2 = 2.77032, y_3 = 0.07426]$

"stable 3-split: ",  $[y_1 = 0.07426, y_2 = -1.55136, y_3 = 2.77032]$

"stable 3-split: ",  $[y_1 = 0.07426, y_2 = 2.77032, y_3 = -1.55136]$

"stable 3-split: ",  $[y_1 = 2.77032, y_2 = -1.55136, y_3 = 0.07426]$

"stable 3-split: ",  $[y_1 = 2.77032, y_2 = 0.07426, y_3 = -1.55136]$

$$1.774445, 81, \begin{bmatrix} 2 & 1 & 0 \\ 9 & 27 & 0 \\ 6 & 36 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.57560, y_2 = 0.15670, y_3 = 2.81343]$

"stable 3-split: ",  $[y_1 = -1.57560, y_2 = 2.81343, y_3 = 0.15670]$

"stable 3-split: ",  $[y_1 = 0.15670, y_2 = -1.57560, y_3 = 2.81343]$

"stable 3-split: ",  $[y_1 = 0.15670, y_2 = 2.81343, y_3 = -1.57560]$

"stable 3-split: ",  $[y_1 = 2.81343, y_2 = -1.57560, y_3 = 0.15670]$

"stable 3-split: ",  $[y_1 = 2.81343, y_2 = 0.15670, y_3 = -1.57560]$

$$2.054595, 75, \begin{bmatrix} 2 & 1 & 0 \\ 6 & 30 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.58270, y_2 = 0.18302, y_3 = 2.82552]$

"stable 3-split: ",  $[y_1 = -1.58270, y_2 = 2.82552, y_3 = 0.18302]$

"stable 3-split: ",  $[y_1 = 0.18302, y_2 = -1.58270, y_3 = 2.82552]$

"stable 3-split: ",  $[y_1 = 0.18302, y_2 = 2.82552, y_3 = -1.58270]$

"stable 3-split: ",  $[y_1 = 2.82552, y_2 = -1.58270, y_3 = 0.18302]$

"stable 3-split: ",  $[y_1 = 2.82552, y_2 = 0.18302, y_3 = -1.58270]$

$$2.130965, 75, \begin{bmatrix} 2 & 1 & 0 \\ 6 & 30 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.59266, y_2 = 0.22195, y_3 = 2.84210]$

"stable 3-split: ",  $[y_1 = -1.59266, y_2 = 2.84210, y_3 = 0.22195]$

"stable 3-split: ",  $[y_1 = 0.22195, y_2 = -1.59266, y_3 = 2.84210]$

"stable 3-split: ",  $[y_1 = 0.22195, y_2 = 2.84210, y_3 = -1.59266]$

"stable 3-split: ",  $[y_1 = 2.84210, y_2 = -1.59266, y_3 = 0.22195]$

"stable 3-split: ",  $[y_1 = 2.84210, y_2 = 0.22195, y_3 = -1.59266]$

$$2.233085, 69, \begin{bmatrix} 2 & 1 & 0 \\ 6 & 24 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

"stable 3-split: ",  $[y_1 = -1.61979, y_2 = 0.34447, y_3 = 2.88547]$

"stable 3-split: ",  $[y_1 = -1.61979, y_2 = 2.88547, y_3 = 0.34447]$

"stable 3-split: ",  $[y_1 = 0.34447, y_2 = -1.61979, y_3 = 2.88547]$

"stable 3-split: ",  $[y_1 = 0.34447, y_2 = 2.88547, y_3 = -1.61979]$

"stable 3-split: ",  $[y_1 = 2.88547, y_2 = -1.61979, y_3 = 0.34447]$

"stable 3-split: ",  $[y_1 = 2.88547, y_2 = 0.34447, y_3 = -1.61979]$

$$2.476580, 63, \begin{bmatrix} 2 & 1 & 0 \\ 6 & 18 & 0 \\ 6 & 30 & 0 \end{bmatrix}$$

$$2.896800, 51, \begin{bmatrix} 2 & 1 & 0 \\ 6 & 18 & 0 \\ 0 & 24 & 0 \end{bmatrix}$$

$$3.860650, 45, \begin{bmatrix} 2 & 1 & 0 \\ 6 & 18 & 0 \\ 0 & 18 & 0 \end{bmatrix}$$

$$5.58724, 39, \begin{bmatrix} 2 & 1 & 0 \\ 3 & 15 & 0 \\ 0 & 18 & 0 \end{bmatrix}$$

(17)

> *add(ArrayNumElems(trajs[i]), i = 1 ..ArrayNumElems(trajs))*

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(18)

> *bifs := map(l→proj12(map(rhs, l)), fsol01rel) :*

*bifsp := pointplot3d(bifs, symbol=solidsphere, color="FireBrick", symbolsize=7) :*

```
> display(bifsp, seq(seq(trajs[i][j], j = 1 ..ArrayNumElems(trajs[i])), i = 1
    ..ArrayNumElems(trajs)),
    labels = [' $\gamma$ ',  $y_1$ ',  $y_2$ '], view = box, orientation = [-85, 80, 0], lightmodel = light4)
```

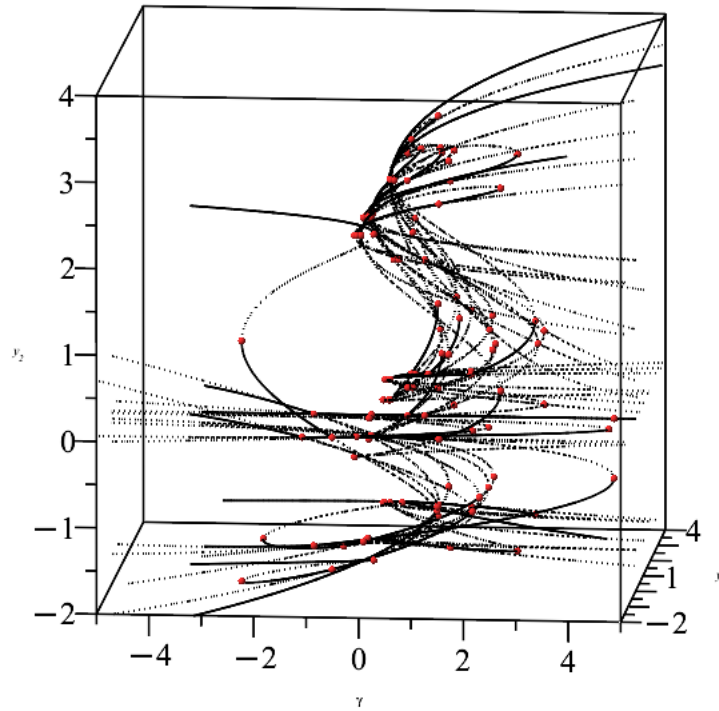


Diagram showing the total number of stationary points (blue), the number of stable stationary points (red) and the number of stable stationary points without symmetry (green)

```
> lth := 3
```

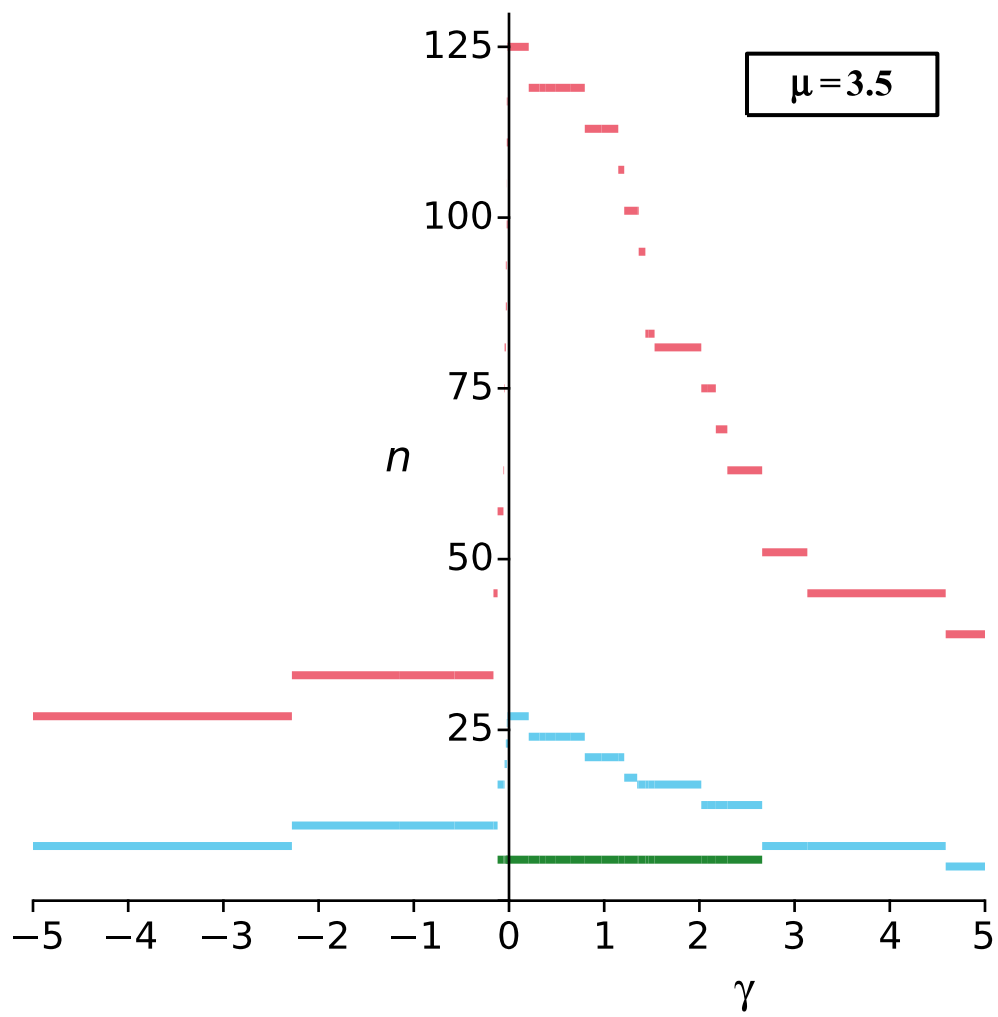
```
lth := 3
```

```
> nsollist := Array(1 ..ngamma + 1) :
  nstablist := Array(1 ..ngamma + 1) :
  nsymlist := Array(1 ..ngamma + 1) :
  nsollist[1] := line([ $\gamma_{\min}$ , nsol[1]], [bifgammarel[1], nsol[1]], color = tolre, thickness = lth) :
  nstablist[1] := line([ $\gamma_{\min}$ , add(sst[1, i, 1], i = 1 ..3)], [bifgammarel[1], add(sst[1, i, 1], i = 1
    ..3)], color = tolcy, thickness = lth) :
  if sst[1, 3, 1]  $\neq$  0 then
    nsymlist[1] := line([ $\gamma_{\min}$ , sst[1, 3, 1]], [bifgammarel[i], sst[1, 3, 1]], color = tolgr, thickness
      = lth) :
  else
    nsymlist[1] := NULL :
  end if:
```

```

nsollist[-1] := line([bifgammarel[-1], nsol[-1]], [ $\gamma_{\max}$ , nsol[-1]], color = tolre, thickness
= lth) :
nstablist[-1] := line([bifgammarel[-1], add(sst[-1, i, 1], i = 1 .. 3)], [ $\gamma_{\max}$ , add(sst[-1, i,
1], i = 1 .. 3)], color = tolcy, thickness = lth) :
if sst[-1, 3, 1]  $\neq$  0 then
    nsymlist[-1] := line([bifgammarel[-1], sst[-1, 3, 1]], [ $\gamma_{\max}$ , sst[-1, 3, 1]], color = tolgr,
    thickness = lth) :
else
    nsymlist[-1] := NULL :
end if:
for i from 2 to ngamma do
    nsollist[i] := line([bifgammarel[i-1], nsol[i]], [bifgammarel[i], nsol[i]], color = tolre,
    thickness = lth) :
    nstablist[i] := line([bifgammarel[i-1], add(sst[i, j, 1], j = 1 .. 3)], [bifgammarel[i],
    add(sst[i, j, 1], j = 1 .. 3)], color = tolcy, thickness = lth) :
    if sst[i, 3, 1]  $\neq$  0 then
        nsymlist[i] := line([bifgammarel[i-1], sst[i, 3, 1]], [bifgammarel[i], sst[i, 3, 1]], color
        = tolgr, thickness = lth) :
    else
        nsymlist[i] := NULL;
    end if;
end do:
display(seq(nsollist[i], i = 1 .. ngamma + 1), seq(nstablist[i], i = 1 .. ngamma + 1),
seq(nsymlist[i], i = 1 .. ngamma + 1),
    textplot([3.5, 120, typeset( $\mu = \text{subs}(mb, \mu)$ ), font = [Helvetica, bold, 14]]),
    rectangle([2.5, 115], [4.5, 124], filled = false),
    labels = [' $\gamma$ ', 'n'], tickmarks = [11, 5], view = [ $\gamma_{\min}$  ..  $\gamma_{\max}$ , 0 .. 130],
    font = [Helvetica, roman, 14], labelfont = [Helvetica, roman, 16])

```



>