> $restart:$

Computation of a classical bifurcation diagram for a second-order coupling via singular ODEs

We use ideas from the Vessiot theory of singular differential equations to derive a vector field such that the bifurcation paths are trajectories and the bifurcation points steady states. In this worksheet, we always have N=M=3; all other parameters can be set in the worksheet. The bifurcation parameter is μ.

> $with(LinearAlgebra) : with(plots) : with(plottools) : with(VectorCalculus) : with(ListTools) : with(RootFinding) : with(Grid) : with(ColorTools) :$

> $interface(warnlevel = 0);$
> $refinelvl := 1;$

$$3$$

$$refinelvl := 1 \tag{1}$$

> $tolbl := Color([68, 119, 170]) :$
> $tolgr := Color([34, 136, 51]) :$
> $tolye := Color([204, 187, 68]) :$
> $tolpu := Color([170, 51, 119]) :$
> $tolre := Color([238, 102, 119]) :$
> $tolcy := Color([102, 204, 238]) :$

Build model.

We set the parameters to their standard values in the paper and use a linear coupling coefficient function.

> $r := 0; o1 := 1; p1 := -1; q1 := 0; o2 := 1; p2 := -2; q2 := 3;$

$$r := 0$$

$$o1 := 1$$

$$p1 := -1$$

$$q1 := 0$$

$$o2 := 1$$

$$p2 := -2$$

$$q2 := 3 \tag{2}$$

> $s := (\mu, x) \rightarrow (o1 \cdot x^2 + p1 \cdot x + q1 - \mu) \cdot (o2 \cdot x^2 + p2 \cdot x + q2 - \mu);$
> $collect(s(\mu, x), x);$

$$s := (\mu, x) \mapsto (o1 \cdot x^2 + p1 \cdot x + q1 + (-\mu)) \cdot (o2 \cdot x^2 + p2 \cdot x + q2 + (-\mu))$$

$$x^4 - 3\,x^3 + (-2\,\mu + 5)\,x^2 + (3\,\mu - 3)\,x - \mu\,(-\mu + 3) \tag{3}$$

> $h := (\beta, \gamma, x) \rightarrow (\beta + \gamma \cdot x)$

$$h := (\beta, \gamma, x) \mapsto \beta + \gamma \cdot x \tag{4}$$

> $\sigma := y_1 + y_2 + y_3$

$$\sigma := y_1 + y_2 + y_3 \tag{5}$$

Define the considered vector field describing the evolution of the traits.

> $f_1 := (r \cdot \mu - y_1) \cdot expand(s(\mu, y_1)) + h(\beta, \gamma, y_1) \cdot \sigma;$

$f_2 := (r \cdot \mu - y_2) \cdot expand(s(\mu, y_2)) + h(\beta, \gamma, y_2) \cdot \sigma;$
$f_3 := (r \cdot \mu - y_3) \cdot expand(s(\mu, y_3)) + h(\beta, \gamma, y_3) \cdot \sigma;$

$$f_1 := -y_1 \left(y_1^4 - 2\mu y_1^2 - 3 y_1^3 + \mu^2 + 3\mu y_1 + 5 y_1^2 - 3\mu - 3 y_1\right) + \left(\gamma y_1 + \beta\right) \left(y_1 + y_2 + y_3\right)$$

$$f_2 := -y_2 \left(y_2^4 - 2\mu y_2^2 - 3 y_2^3 + \mu^2 + 3\mu y_2 + 5 y_2^2 - 3\mu - 3 y_2\right) + \left(\gamma y_2 + \beta\right) \left(y_1 + y_2 + y_3\right)$$

$$f_3 := -y_3 \left(y_3^4 - 2\mu y_3^2 - 3 y_3^3 + \mu^2 + 3\mu y_3 + 5 y_3^2 - 3\mu - 3 y_3\right) + \left(\gamma y_3 + \beta\right) \left(y_1 + y_2 + y_3\right) \quad \textbf{(6)}$$

Compute its Jacobians both with respect to the variables only and in addition with respect to the environmental parameter μ.

> $J := simplify(Jacobian([f_1, f_2, f_3], [y_1, y_2, y_3]));$
> $JJ := simplify(Jacobian([f_1, f_2, f_3], [\mu, y_1, y_2, y_3]));$

$J :=$

$$\begin{bmatrix} -5 y_1^4 + 12 y_1^3 + 3(-5 + 2\mu) y_1^2 + 2(3 - 3\mu + \gamma) y_1 - \mu^2 - \cdots \\ \gamma y_2 + \beta \quad \cdots \\ \gamma y_3 + \beta \quad \cdots \end{bmatrix}$$

$JJ :=$ **(7)**

$$\begin{bmatrix} 2 y_1^3 - 3 y_1^2 + (-2\mu + 3) y_1 & -5 y_1^4 + 12 y \cdots \\ 2 y_2^3 - 3 y_2^2 + (-2\mu + 3) y_2 & \cdots \\ 2 y_3^3 - 3 y_3^2 + (-2\mu + 3) y_3 & \cdots \end{bmatrix}$$

> $detJ := Determinant(J):$

Setting up vector field **Y** generating the **projected Vessiot distribution** using the adjoint of the Jacobian and the Jacobian with respect to μ alone.

> $C := Adjoint(J):$

> $M := Jacobian([f_1, f_2, f_3], [\mu]);$

(8)

$$M := \begin{bmatrix} -y_1\left(-2\,y_1^2+2\,\mu+3\,y_1-3\right) \\ -y_2\left(-2\,y_2^2+2\,\mu+3\,y_2-3\right) \\ -y_3\left(-2\,y_3^2+2\,\mu+3\,y_3-3\right) \end{bmatrix} \tag{8}$$

> $b := -expand(C \cdot M)$:
> $Y := [detJ, b[1,1], b[2,1], b[3,1]]$:

We consider now only the case β=0 and γ=1, i.e. pure second-order coupling.

> $bg01 := \beta = 0, \gamma = 1;$

$$bg01 := \beta = 0, \gamma = 1 \tag{9}$$

> $f01_1 := subs(bg01, f_1) : f01_2 := subs(bg01, f_2) : f01_3 := subs(bg01, f_3)$:
  $J01 := subs(bg01, J) : JJ01 := subs(bg01, JJ) : detJ01 := subs(bg01, detJ)$:
  $Y01 := subs(bg01, Y) : b01 := subs(bg01, b)$:

We use **Y** to set up a **system** in a form suitable for integration with **dsolve**.

> $vars := [\mu(t), y_1(t), y_2(t), y_3(t)]$:
  $trafo := \mu = \mu(t), y_1 = y_1(t), y_2 = y_2(t), y_3 = y_3(t)$:

> $Y01t := subs(trafo, Y01)$:
  $sysY01 := [diff(\mu(t), t) = Y01t[1], diff(y_1(t), t) = Y01t[2], diff(y_2(t), t) = Y01t[3], diff(y_3(t), t) = Y01t[4]]$:
  $sysY01m := [diff(\mu(t), t) = -Y01t[1], diff(y_1(t), t) = -Y01t[2], diff(y_2(t), t) = -Y01t[3], diff(y_3(t), t) = -Y01t[4]]$:

Compute numerically bifurcation points (requires larger number of digits for better identification of identical μ-values).

> $sys01 := [f01_1, f01_2, f01_3, detJ01]$:

> $st := time[real]() : fsol01 := Isolate(sys01, [\mu, y_1, y_2, y_3], digits = 15) : time[real]() - st;$
  $nops(fsol01);$

$$66.753$$
$$141 \tag{10}$$

We store the found bifurcation points in a **data file**.

> fname := FileTools:-JoinPath(["Maple", "MathBiol", "Speciation" , cat("BifPointsN3M3G", sprintf("%4.2f", subs(bg01, γ)), ".txt")], base = homedir);
  fd := fopen(fname, WRITE) :
  fprintf(fd, "Bifurcation points computed by BifDia2DN3M3G1-Grid.mw\n") :
  fprintf(fd, "o1=%.1f, p1=%.1f, q1=%.1f, o2=%.1f, p2=%.1f, q2=%.1f, r=%.3f, N=%1d, gamma=%4.2f\n", o1, p1, q1, o2, p2, q2, r, 3, subs(bg01, γ)) :
  fprintf(fd, "mu\t\t x1\t\t x2\t\t x3\n") :

fname := "C:\Users\seiler\Maple\MathBiol\Speciation\BifPointsN3M3G1.00.txt" **(11)**

```
> for bf in fsol01 do
    fprintf( fd, "%+12.8f\t %+12.8f\t %+12.8f\t %+12.8f\n", op(map(rhs, bf) ) ) :
  end do:
>  fclose(fd);
```

We select the μ-values of the bifurcation points and select those in the specified range.

```
> bifmu := MakeUnique(map(l→rhs(l[1]), fsol01)); nops(bifmu)
```

$bifmu := [-1., -0.847953364593873, -0.824555320336759, -0.743790752804816, -0.654297355697598, -0.642253003193784, -0.482757922275626, -0.337212972879509, 0., 1.60574282268168, 1.95903100881521, 2.19358930436987, 2.20958168501405, 2.31246704183670, 2.35678153849504, 2.44794105507108, 2.57554312315334, 2.58412334400038, 2.66583611547788, 2.73051766553203, 2.75000000000000, 2.88445820031030, 2.89899672284716, 3., 3.12422878516019, 3.16430007065674, 3.29867639032435, 3.34695711494097, 3.34809135082230, 3.51280679956613, 3.76428477573808, 3.80206526303883, 3.89530340246740, 4.15366991669183, 4.44377927280554, 6.05588513685381, 11.8245553203368, 12.0197378398219, 12.4316359140501, 15., 17.6938040544195, 17.8510032771528, 30.5806002348563, 39.9799945584335, 46.6299412880543]$

$$45 \tag{12}$$

```
> μ_min := -1; μ_max := 7;
  bifmurel := select(x→(x > μ_min) and (x < μ_max), bifmu); nops(bifmurel);
  fsol01rel := select(x→rhs(x[1]) < μ_max, fsol01) :
  bfpcolor := [ 0 $ nops(fsol01rel) ] :
```

$$\mu_{min} := -1$$

$$\mu_{max} := 7$$

$bifmurel := [-0.847953364593873, -0.824555320336759, -0.743790752804816, -0.654297355697598, -0.642253003193784, -0.482757922275626, -0.337212972879509, 0., 1.60574282268168, 1.95903100881521, 2.19358930436987, 2.20958168501405, 2.31246704183670, 2.35678153849504, 2.44794105507108, 2.57554312315334, 2.58412334400038, 2.66583611547788, 2.73051766553203, 2.75000000000000, 2.88445820031030, 2.89899672284716, 3., 3.12422878516019, 3.16430007065674, 3.29867639032435, 3.34695711494097, 3.34809135082230, 3.51280679956613, 3.76428477573808, 3.80206526303883, 3.89530340246740, 4.15366991669183, 4.44377927280554, 6.05588513685381]$

$$35 \tag{13}$$

Some auxiliary quantities for the plotting.

```
> proj12 := l→[l[1], Norm(Vector(l[2..4]))] :
  projsig := l→[l[1], l[2] + l[3] + l[4]] :
  round7 := x→1e-7·round(1e7·x) :
```

```
box := [-1..7, -0.5..7];
maxbox := [-2..300, -20..20];
tend := 100;
```

$$box := [-1..7, -0.5..7]$$
$$maxbox := [-2..300, -20..20]$$
$$tend := 100 \quad \textbf{(14)}$$

We generate a list of μ-values lying before, after and between the chosen bifurcation values. The steady states at this μ-values are used as initial points for computing pieces of the bifurcation paths.

```
> nmu := nops(bifmurel);
  mulist := [ 0 $ nmu + 1 ]:
  checkmulist := [ 0 $ nmu + 1 ]:
```

$$nmu := 35 \quad \textbf{(15)}$$

```
> mulist[1] := bifmurel[1] - 1 :
  mulist[-1] := bifmurel[-1] + 1 :
> for i from 1 to nmu - 1 do
     mulist[i + 1] := 0.5·(bifmurel[i] + bifmurel[i + 1]);
  end do:
  checkmulist := [op(map(m→[μ(t) - m, halt], mulist)), [μ(t) - μ_max, halt]]:
```

The following procedure integrates Y along the i-th segment of the μ-list. For plotting the results in 2D, we use either the distance from the origin or the value of σ as ordinate.

```
> bifsegment := proc(i)
  local fsolm, ivps, intrange, checkmui, j, Jj, revsj, evsjm, evsjM, tsty, tcol, nvals, dsn, trajs, sst;
  global nmu, tend, mulist, checkmulist, maxbox, refinelvl, J01, f01_1, f01_2, f01_3, trafo;
  fsolm := Isolate(subs(μ = mulist[i], [f01_1, f01_2, f01_3]), [y_1, y_2, y_3], Digits = 15);
  ivps := map(l→[μ(0) = mulist[i], op(l)], map(l→subs(trafo, t = 0, l), fsolm));
  if i = 1 then
     intrange := -tend..0;
  elif i = nmu + 1 then
     intrange := 0..tend;
  else
     intrange := -tend..tend;
  end if;
  trajs := Array(1..nops(fsolm), 0);
  sigtrajs := Array(1..nops(fsolm), 0);
  sst := Array(1..7, 0);
  checkmui := subsop(i = NULL, checkmulist);
  for j from 1 to nops(fsolm) do
     Jj := subs(op(ivps[j]), subs(trafo, t = 0, J01));
     revsj := map(Re, Eigenvalues(Jj));
     if mul(revsj) = 0 then
        tcol := "Black";
        tsty := solid;
        sst[6] := sst[6] + 1;
     end if;
     evsjM := max(revsj);
```

```
        evsjm := min(revsj);
        nvals := nops({op(map(round7, map(rhs, fsolm[j])))});
        sst[nvals] := sst[nvals] + 1;
        if evsjM < 0 then
          tsty := solid;
          if nvals = 3 then
            tcol := tolgr;
            sst[7] := sst[7] + 1;
          else
            tcol := tolcy;
          end if;
          sst[4] := sst[4] + 1;
        elif evsjm > 0 then
          tcol := tolpu;
          tsty := dot;
          sst[5] := sst[5] + 1;
        elif evsjm·evsjM < 0 then
          tcol := tolye;
          tsty := dot;
          sst[5] := sst[5] + 1;
        else error "not yet handled";
        end if;
        dsn := dsolve([op(sysY01), op(ivps[j])], numeric, vars, method = rkf45, range = intrange,
         events = [[Y01t[1], halt], op(checkmui)]) :
        trajs[j] := odeplot(dsn, [μ(t), Norm(Vector([y_1(t), y_2(t), y_3(t)]))], color = tcol, linestyle
          = tsty, thickness = 1, refine = refinelvl, view = maxbox) :
        sigtrajs[j] := odeplot(dsn, [μ(t), y_1(t) + y_2(t) + y_3(t)], color = tcol, linestyle = tsty, thickness
          = 1, refine = refinelvl, view = maxbox) :
      end do:
      [i, nops(fsolm), trajs, sigtrajs, sst];
      end proc:
> Set('nmu','tend','mulist','checkmulist','maxbox','refinelvl','sysY01','Y01t','f01_1','f01_2','f01_3','J01','vars
     ','round7','trafo','tolbl','tolgr','tolye','tolpu','tolre','tolcy') :
> st := time[real]( ) : BifSegs := Map(bifsegment, [ii $ ii = 1 ..nmu + 1]) : time[real]( ) − st;
```

247.602 **(16)**

```
> trajs := Array(1 ..nmu + 1, 0) :
  sigtrajs := Array(1 ..nmu + 1, 0) :
  sst := Array(1 ..nmu + 1, 1 ..7, 0) :
  for i from 1 to nmu + 1 do
    print(BifSegs[i, 1], mulist[BifSegs[i, 1]], BifSegs[i, 2], BifSegs[i, 5]);
    trajs[i] := BifSegs[i, 3] :
    sigtrajs[i] := BifSegs[i, 4] :
    sst[i] := BifSegs[i, 5] :
  end do:
```

$$1, -1.847953365, 1, \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$2, -0.8362543425, 9, \begin{bmatrix} 3 & 6 & 0 & 2 & 7 & 0 & 0 \end{bmatrix}$

$3, -0.7841730365, 9, \begin{bmatrix} 3 & 6 & 0 & 2 & 7 & 0 & 0 \end{bmatrix}$

$4, -0.6990440540, 15, \begin{bmatrix} 3 & 12 & 0 & 5 & 10 & 0 & 0 \end{bmatrix}$

$5, -0.6482751795, 15, \begin{bmatrix} 3 & 12 & 0 & 5 & 10 & 0 & 0 \end{bmatrix}$

$6, -0.5625054630, 21, \begin{bmatrix} 3 & 12 & 6 & 5 & 16 & 0 & 0 \end{bmatrix}$

$7, -0.4099854476, 27, \begin{bmatrix} 3 & 18 & 6 & 8 & 19 & 0 & 0 \end{bmatrix}$

$8, -0.1686064864, 27, \begin{bmatrix} 3 & 18 & 6 & 8 & 19 & 0 & 0 \end{bmatrix}$

$9, 0.8028714115, 27, \begin{bmatrix} 3 & 18 & 6 & 8 & 19 & 0 & 0 \end{bmatrix}$

$10, 1.782386916, 33, \begin{bmatrix} 3 & 24 & 6 & 11 & 22 & 0 & 0 \end{bmatrix}$

$11, 2.076310156, 45, \begin{bmatrix} 3 & 24 & 18 & 11 & 34 & 0 & 0 \end{bmatrix}$

$12, 2.201585494, 45, \begin{bmatrix} 3 & 24 & 18 & 11 & 34 & 0 & 0 \end{bmatrix}$

$13, 2.261024364, 51, \begin{bmatrix} 3 & 30 & 18 & 14 & 37 & 0 & 0 \end{bmatrix}$

$14, 2.334624290, 57, \begin{bmatrix} 3 & 30 & 24 & 14 & 43 & 0 & 0 \end{bmatrix}$

$15, 2.402361296, 63, \begin{bmatrix} 3 & 36 & 24 & 14 & 49 & 0 & 0 \end{bmatrix}$

$16, 2.511742089, 57, \begin{bmatrix} 3 & 30 & 24 & 11 & 46 & 0 & 0 \end{bmatrix}$

$17, 2.579833234, 57, \begin{bmatrix} 3 & 30 & 24 & 11 & 46 & 0 & 0 \end{bmatrix}$

$18, 2.624979730, 63, \begin{bmatrix} 3 & 36 & 24 & 11 & 52 & 0 & 0 \end{bmatrix}$

$19, 2.698176890, 69, \begin{bmatrix} 3 & 36 & 30 & 11 & 58 & 0 & 0 \end{bmatrix}$

$20, 2.740258833, 81, \begin{bmatrix} 3 & 36 & 42 & 17 & 64 & 0 & 6 \end{bmatrix}$

$21, 2.817229100, 83, \begin{bmatrix} 5 & 36 & 42 & 18 & 65 & 0 & 6 \end{bmatrix}$

$$22, 2.891727462, 89, \begin{bmatrix} 5 & 42 & 42 & 18 & 71 & 0 & 6 \end{bmatrix}$$

$$23, 2.949498362, 89, \begin{bmatrix} 5 & 42 & 42 & 18 & 71 & 0 & 6 \end{bmatrix}$$

$$24, 3.062114392, 89, \begin{bmatrix} 5 & 42 & 42 & 18 & 71 & 0 & 6 \end{bmatrix}$$

$$25, 3.144264428, 101, \begin{bmatrix} 5 & 42 & 54 & 18 & 83 & 0 & 6 \end{bmatrix}$$

$$26, 3.231488230, 101, \begin{bmatrix} 5 & 42 & 54 & 18 & 83 & 0 & 6 \end{bmatrix}$$

$$27, 3.322816752, 107, \begin{bmatrix} 5 & 48 & 54 & 21 & 86 & 0 & 6 \end{bmatrix}$$

$$28, 3.347524233, 107, \begin{bmatrix} 5 & 48 & 54 & 21 & 86 & 0 & 6 \end{bmatrix}$$

$$29, 3.430449076, 113, \begin{bmatrix} 5 & 48 & 60 & 21 & 92 & 0 & 6 \end{bmatrix}$$

$$30, 3.638545788, 113, \begin{bmatrix} 5 & 48 & 60 & 21 & 92 & 0 & 6 \end{bmatrix}$$

$$31, 3.783175020, 113, \begin{bmatrix} 5 & 48 & 60 & 21 & 92 & 0 & 6 \end{bmatrix}$$

$$32, 3.848684332, 119, \begin{bmatrix} 5 & 54 & 60 & 24 & 95 & 0 & 6 \end{bmatrix}$$

$$33, 4.024486660, 119, \begin{bmatrix} 5 & 54 & 60 & 24 & 95 & 0 & 6 \end{bmatrix}$$

$$34, 4.298724595, 119, \begin{bmatrix} 5 & 54 & 60 & 24 & 95 & 0 & 6 \end{bmatrix}$$

$$35, 5.249832205, 119, \begin{bmatrix} 5 & 54 & 60 & 24 & 95 & 0 & 6 \end{bmatrix}$$

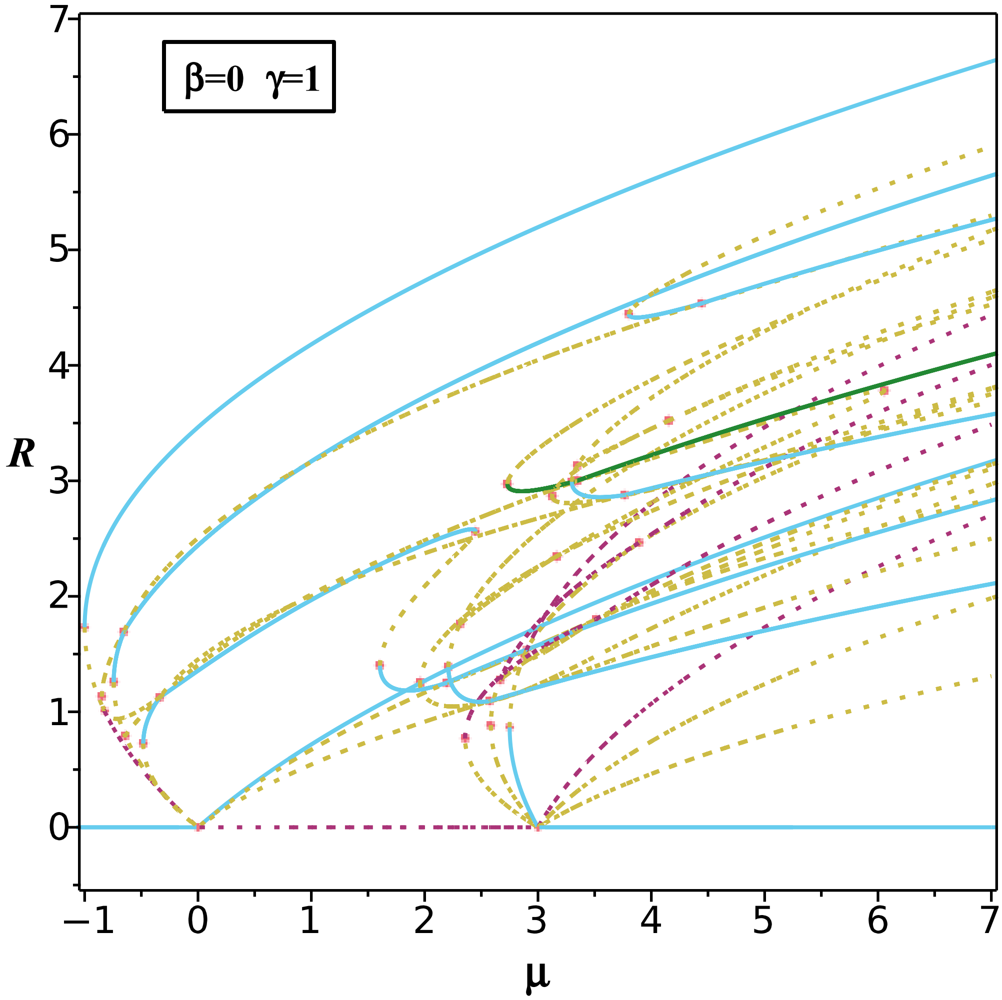
$$36, 7.055885137, 107, \begin{bmatrix} 5 & 54 & 48 & 24 & 83 & 0 & 6 \end{bmatrix} \quad \textbf{(17)}$$

```
> add(ArrayNumElems(trajs[i]), i = 1 ..ArrayNumElems(trajs))
```

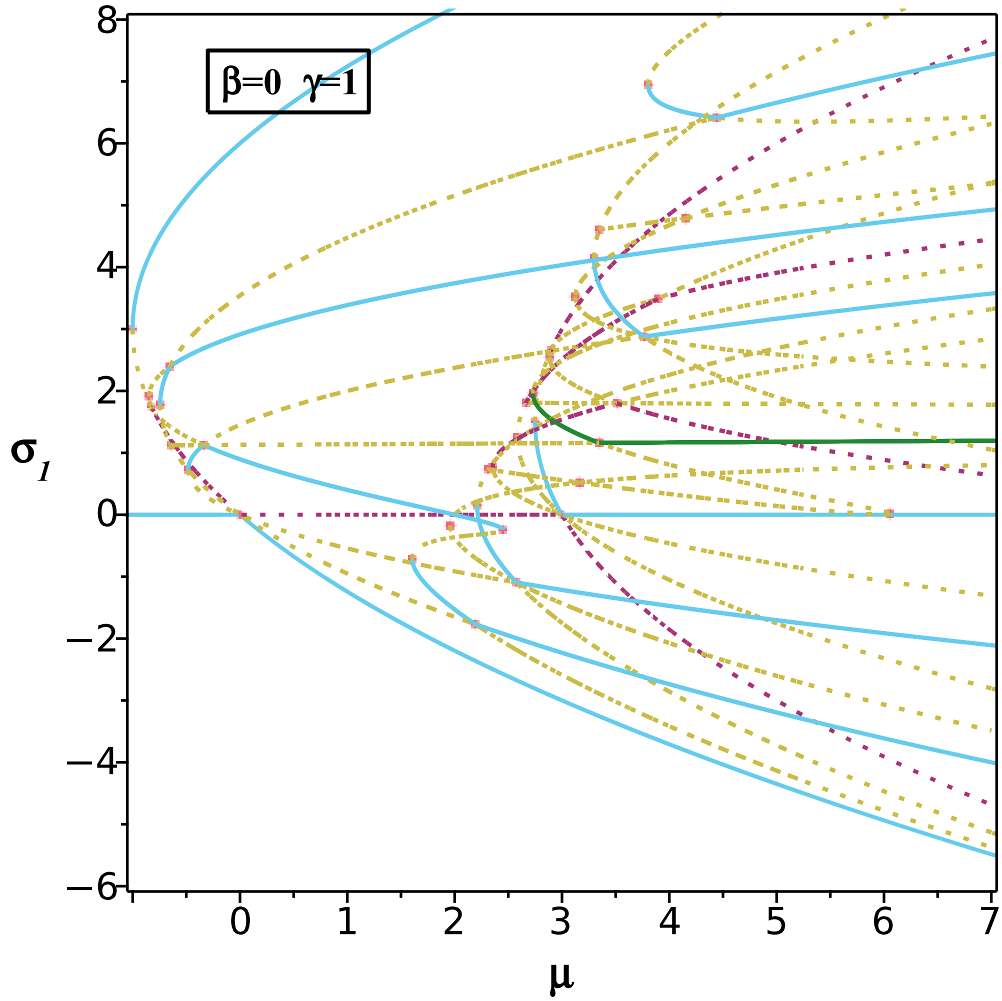
$$2460 \quad \textbf{(18)}$$

```
> bifs := map(l→proj12(map(rhs, l)), fsol01rel) :
  bifsp := pointplot(bifs, symbol = soliddiamond, color = tolre, symbolsize = 7) :
> bifsig := map(l→projsig(map(rhs, l)), fsol01rel) :
  bifsigp := pointplot(bifsig, symbol = soliddiamond, color = tolre, symbolsize = 7) :
> display(bifsp, seq(seq(trajs[i][j], j = 1 ..ArrayNumElems(trajs[i])), i = 1
     ..ArrayNumElems(trajs)),
         textplot([0.5, 6.5, typeset("β=", subs(bg01, β), " γ=", subs(bg01, γ)), font
     = [Helevetica, bold, 14]]),
```

rectangle([−0.3, 6.2], [1.2, 6.8], *filled* = *false*), *font* = [*Helvetica*, *roman*, 14],
labels = ['μ','*R*'], *labelfont* = [*Helevetica*, *bold*, 16], *view* = *box*, *axes* = *boxed*)



> *display*(*bifsigp*, *seq*(*seq*(*sigtrajs*[*i*][*j*], *j* = 1 ..*ArrayNumElems*(*trajs*[*i*])), *i* = 1 ..*ArrayNumElems*(*trajs*)),
textplot([0.5, 7, *typeset*("β=", *subs*(*bg01*, β), " γ=", *subs*(*bg01*, γ)), *font* = [*Helevetica*, *bold*, 14]]),
rectangle([−0.3, 6.5], [1.2, 7.5], *filled* = *false*), *font* = [*Helvetica*, *roman*, 14],
labels = ['μ','σ_1'], *labelfont* = [*Helevetica*, *bold*, 16], *view* = [−1 ..7, −6 ..8], *axes* = *boxed*)


β=0 γ=1
8
6
4
2
0
−2
−4
−6
σ_1
0
1
2
3
4
5
6
7
μ

>